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Mathematics I

Differential and Integral Calculus

Strategic project of TBU in Zlín, reg. no. CZ.02.2.69/0.0/0.0/16_015/0002204



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1. Real Functions

Functions enables us to describe the real world problems in mathematical terms. The body volume varies with temperature, the interest paid on a cash investment depends on the length of time the investment is held. If the phenomenon under consideration depends on a single variable whose values are real, we are talking about the function of one real variable.



1.1 Definition of a Function

Definition 1.1 (“Math Tutor”, 2019)

By a **real function of a real variable** we mean any mapping from some subset of the set of real numbers to the set of real numbers.

This is the precise definition of a real function of a real variable. Without knowing anything about mappings this definition is not suitable for practical use. Therefore, we will explain a real functions in another, more applicable, way.

Definition 1.2 (Mařík, 2012)

Let A and B be nonempty sets of real numbers. Let f be a rule which associates each element x of the set A with exactly one element y of the set B . The rule f is said to be a **function** defined on A . We write $f : A \rightarrow B$. If f associates x with y , we write $y = f(x)$.

The variable x is called an **independent variable** and y a **dependent variable**.

The set A is called a **domain** of the function f and denoted by $D(f)$ (or D_f). It is the set of all numbers x that can be substituted into the function, that is, for which the formula defining the function makes sense.



If we define a function $y = f(x)$ by a formula without specifying the domain, then the domain is assumed to be the largest set of all real numbers x for which the formula defining the function makes sense.

The set of all values of $f(x)$ for $x \in D_f$ is called the **range** of the function f . It is denoted R_f (or $\text{ran}(f)$). The range R_f is the subset of the set B . We can write

$$R_f = \{f(x) : x \in D_f\}.$$

When the range of a function is a set of real numbers, the function is said to be **real-valued**.

Example 1.3

Determine the domain of the function $f(x) = \frac{x-1}{x^2-3x-10}$.

Solution The domain for this function are all the values x for which we do not have division by zero. We need set the denominator equal to zero and solve.

$$x^2 - 3x - 10 = 0 \quad \Rightarrow \quad (x-5)(x+2) = 0 \quad \Rightarrow \quad x = 5, x = -2.$$

We will get division by zero if we plug in $x = -2$ or $x = 5$. So, the domain is $D_f = (-\infty, -2) \cup (-2, 5) \cup (5, \infty)$.



Example 1.4

Determine the domain of the function $f(x) = \sqrt{7 - 4x}$.

Solution We cannot take the square root of a negative number, it is required $7 - 4x \geq 0$. We solve this inequality

$$7 - 4x \geq 0 \quad \Rightarrow \quad x \leq \frac{7}{4}$$

and the domain is $D_f = (-\infty, \frac{7}{4}]$.

.....
To visualize functions we draw its **graph**.

Definition 1.5 (Mařík, 2012)

Let f be a function. A **graph** of the function f is the set of all of the points in the plane $(x, y) \in \mathbb{R}^2$ with the property $y = f(x)$:

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} : x \in D_f \wedge y = f(x)\}.$$



1.2 Operations with Functions

We can perform basic algebraic operations with functions. If f and g are functions, then for all $x \in D_f \cap D_g$ we define **addition** $f + g$, **subtraction** $f - g$, **multiplication** $f \cdot g$, **division** f/g by the formulas

$$(f + g)(x) = f(x) + g(x),$$

$$(f - g)(x) = f(x) - g(x),$$

$$(f \cdot g)(x) = f(x) \cdot g(x),$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad \text{where } g(x) \neq 0.$$

The next operation is called the **composition** of the functions. For given functions f and g the **composite** function $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x)).$$

The function f is said to be an **outside** function and g an **inside** function. The domain of $f \circ g$ is the set of all $x \in D_g$ for which $g(x) \in D_f$.



Example 1.6

For given $f(x) = x^2 + 1$ and $g(x) = x + 3$ find $f \pm g, f \cdot g, f/g, f \circ g, g \circ f, f \circ f, g \circ g$.

Solution We will perform basic algebraic operations with functions f and g and simplify them as much as possible.

$$(f + g)(x) = f(x) + g(x) = x^2 + 1 + x + 3 = x^2 + x + 4,$$

$$(f - g)(x) = f(x) - g(x) = x^2 + 1 - (x + 3) = x^2 - x - 2,$$

$$(f \cdot g)(x) = f(x) \cdot g(x) = (x^2 + 1)(x + 3) = x^3 + 3x^2 + x + 3,$$

$$(f/g)(x) = \frac{f(x)}{g(x)} = \frac{x^2 + 1}{x + 3}, x \neq -3,$$

$$(f \circ g)(x) = f(g(x)) = f(x + 3) = (x + 3)^2 + 1 = x^2 + 6x + 10,$$

$$(g \circ f)(x) = g(f(x)) = g(x^2 + 1) = x^2 + 1 + 3 = x^2 + 4,$$

$$(f \circ f)(x) = f(f(x)) = f(x^2 + 1) = (x^2 + 1)^2 + 1 = x^4 + 2x^2 + 2,$$

$$(g \circ g)(x) = g(g(x)) = g(x + 3) = x + 3 + 3 = x + 6.$$



1.3 Basic Properties of Functions

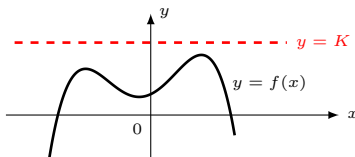
Definition 1.7 (Bouchala & Sadowská, 2007)

Let $M \subset D_f$. A function f is said to be **bounded above on the set M** if a set

$$f(M) := \{f(x) : x \in M\}$$

is bounded above. A function f is said to be **bounded above** if it is bounded above on D_f . **Below-bounded** functions and **bounded** functions are defined analogously.

Equivalently, a real function f is bounded from above if there is a number K such that for all x from the domain D_f one has $f(x) \leq K$. It means that there exists a horizontal line such that the graph of the function bounded above lies below this line.



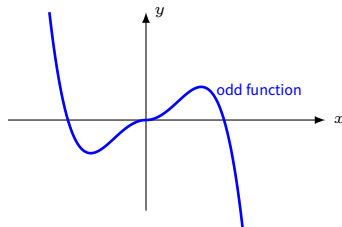
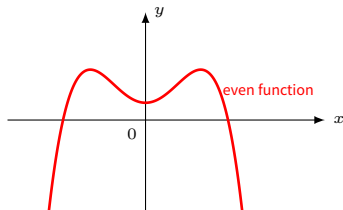


Definition 1.8 (Bouchala & Sadowská, 2007)

A function f is called

- ▶ **even** if $\forall x \in D_f : f(-x) = f(x)$,
- ▶ **odd** if $\forall x \in D_f : f(-x) = -f(x)$.

If f is even or odd, then $\forall x \in D_f : -x \in D_f$. The graph of an even function is symmetric about the y -axis. The graph of an odd function is symmetric about the origin. The function is said to **have a parity** if it is either even or odd.

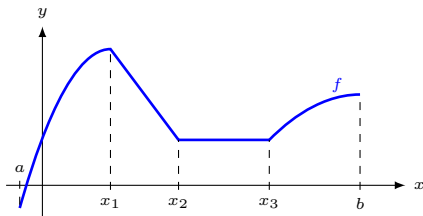




Definition 1.9 (Bouchala & Sadowská, 2007)

Let $M \subset D_f$. A function f is said to be

- ▶ **increasing on the set M** if $\forall x_1, x_2 \in M : x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$,
- ▶ **decreasing on the set M** if $\forall x_1, x_2 \in M : x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$,
- ▶ **non-increasing on the set M** if $\forall x_1, x_2 \in M : x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$,
- ▶ **non-decreasing on the set M** if $\forall x_1, x_2 \in M : x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$.



The function f shown in the graph is:

- ▶ increasing on $[a, x_1], [x_3, b]$,
- ▶ decreasing on $[x_1, x_2]$,
- ▶ non-increasing on $[x_1, x_3]$,
- ▶ non-decreasing on $[x_2, b]$,
- ▶ both non-increasing and non-decreasing on $[x_2, x_3]$.



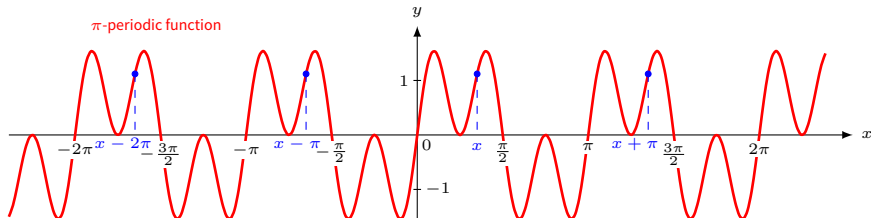
Definition 1.10 (Hass, Giordano, Weir & Thomas, 2005)

A function f is said to be **periodic** if there exists a $p > 0$ such that

$$\forall x \in D_f : f(x) = f(x + p).$$

The smallest such p is called the **period** of f .

If f is a periodic function with a period p , then f is periodic also with period $n \cdot p$ for $n \in \mathbb{N}$.



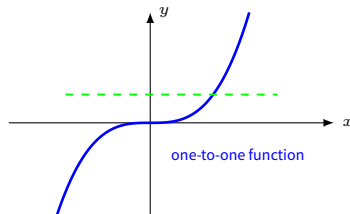
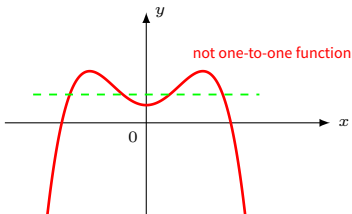


Definition 1.11 (Dawkins, 2018)

A function f is said to be **one-to-one** (or **injective**) if

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

The two different values x_1, x_2 cannot have the same value y . An arbitrary horizontal line can intersect the graph of a one-to-one function at most once.





Definition 1.12 (Stitz & Zeager, 2013)

Suppose f and g are two functions such that

1. $f(g(x)) = x$ for all x in the domain of g and
2. $g(f(x)) = x$ for all x in the domain of f ,

then f and g are **inverses** of each other and the functions f and g are said to be **invertible**. Specifically, we will say that g is an inverse function of f and denote it by f^{-1} .

The “ -1 ” in the notation f^{-1} (read f -inverse) is not an exponent, so we have to remember that

$$f^{-1}(x) \neq \frac{1}{f(x)}.$$

Theorem 1.13 (Bouchala & Sadowská, 2007)

Let f be a function. Then f^{-1} exists if and only if f is injective.

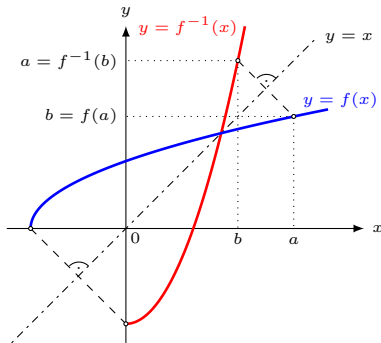
The statement “ f is invertible” is equivalent to “ f is one-to-one”. An increasing or decreasing function on an interval is one-to-one and therefore invertible.



There exists exactly one inverse function of f .

The range of f is the domain of f^{-1} and the domain of f is the range of f^{-1} .

A point $(a; b)$ is on the graph of f if and only if $(b; a)$ is on the graph of f^{-1} . The graphs of f and f^{-1} are symmetric about the line $y = x$.





We can find the inverse function in the following steps.

1. Determine whether the given function is one-to-one.
2. Solve the equation $y = f(x)$ for x .
3. Interchange x and y to get a formula $y = f^{-1}(x)$.

Example 1.14

Find the inverse function of the function $f(x) = \frac{1-3x}{7}$.

Solution To determine if f is one-to-one function we take the equation $f(x_1) = f(x_2)$ and check whether $x_1 = x_2$.

$$f(x_1) = f(x_2)$$

$$\frac{1-3x_1}{7} = \frac{1-3x_2}{7}$$

$$1 - 3x_1 = 1 - 3x_2$$

$$-3x_1 = -3x_2$$

$$x_1 = x_2$$

Hence, the function f is one-to-one, so it is invertible.



Now, we solve the equation $y = f(x)$ for x .

$$y = \frac{1 - 3x}{7}$$

$$7y = 1 - 3x$$

$$7y - 1 = -3x$$

$$\frac{-7y + 1}{3} = x$$

$$\frac{-7x + 1}{3} = y$$

Form an equation.

Multiply both sides by 7.

Subtract 1 from both sides.

Divide both sides by -3 .

Interchange x and y .

We have $y = f^{-1}(x) = \frac{-7x + 1}{3}$. We can verify our result by checking that $(f \circ f^{-1})(x) = x$ for all x in the range of f which is the set of real numbers.

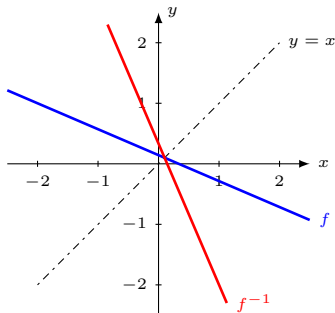
$$(f^{-1} \circ f)(x) = \frac{-7 \cdot \frac{1-3x}{7} + 1}{3} = \frac{-1 + 3x + 1}{3} = \frac{3x}{3} = x.$$



Further, $(f^{-1} \circ f)(x) = x$ must be true for all $x \in D_f$, in our case $D_f = \mathbb{R}$.

$$(f \circ f^{-1})(x) = \frac{1 - 3 \cdot \frac{-7x+1}{3}}{7} = \frac{1 + 7x - 1}{7} = \frac{7x}{7} = x.$$

We can also verify our result graphically. Here is the graph of the function f and its inverse function. Their graphs are symmetric about the line $y = x$.





Example 1.15

Find the inverse function of the function $f(x) = \frac{3}{x-4} + 2$.

Solution The domain of f is $D_f = \mathbb{R} - \{4\}$. We determine if f is one-to-one function.

$$\begin{aligned}\frac{3}{x_1 - 4} + 2 &= \frac{3}{x_2 - 4} + 2 \\ \frac{3}{x_1 - 4} &= \frac{3}{x_2 - 4} \\ x_2 - 4 &= x_1 - 4 \\ x_1 &= x_2\end{aligned}$$

The function f is one-to-one function. We find the function f^{-1} .

$$\begin{aligned}y &= \frac{3}{x-4} + 2 \\ y - 2 &= \frac{3}{x-4}\end{aligned}$$

Form an equation.

Subtract 2 from both sides.



$$x - 4 = \frac{3}{y - 2}$$

Multiply both sides by $x - 4$ and divide by $y - 2$.

$$x = \frac{3}{y - 2} + 4$$

Add 4 to both sides.

$$y = \frac{3}{x - 2} + 4$$

Interchange x and y .

We have $y = f^{-1}(x) = \frac{3}{x - 2} + 4$.

Example 1.16

Find the inverse function of the function $f(x) = x^2 - 1$.

Solution The function f is defined for all $x \in \mathbb{R}$. This function is not one-to-one, because $x_1 = -1$ and $x_2 = 1$ give $f(x_1) = f(x_2) = 0$, for example. So, it has no inverse. Sometimes, if a function is not one-to-one, we restrict the domain of f to a set on which a function f is one-to-one. Let us consider the restricted domain to be $[0, \infty)$. Then we can find an inverse as usual.



$$y = x^2 - 1, x \geq 0$$

$$y + 1 = x^2$$

$$\sqrt{y + 1} = \sqrt{x^2}$$

$$\sqrt{y + 1} = |x| = x$$

We take $|x| = x$ because $x \geq 0$. Now, we switch x and y to get the inverse function $f^{-1}(x) = \sqrt{x + 1}$. The range of the inverse is now equal to the restricted domain of the original function. We can verify that both composition lead to the identity function.

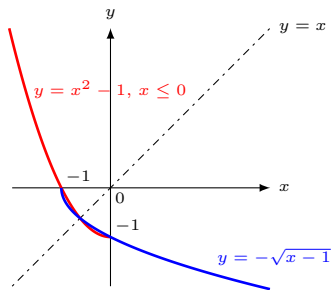
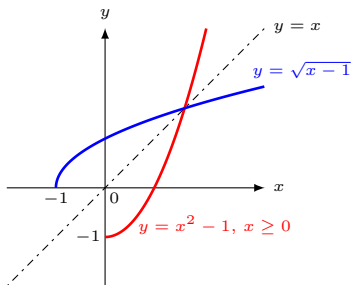
$$f^{-1}(f(x)) = f^{-1}(x^2 - 1) = \sqrt{x^2 - 1 + 1} = |x| = x \quad \text{for } x \geq 0$$

$$f(f^{-1}(x)) = f(\sqrt{x + 1}) = (\sqrt{x + 1})^2 - 1 = x + 1 - 1 = x.$$

If you choose the restricted domain to be $(-\infty, 0]$, the inverse is $f^{-1}(x) = -\sqrt{x + 1}$.



The graphs of f for $x \geq 0$ and its inverse show that they are reflections about the line $y = x$. Similar situation is for f if $x \leq 0$.





1.4 Elementary Functions

Polynomial functions, rational functions, exponential functions, logarithmic functions, trigonometric functions and their inverse functions are called **basic elementary functions**.

A function built up of a finite combination (it means addition, subtraction, multiplication, division, composition of functions) of basic elementary functions is called an **elementary function**. These functions are very important and will frequently encountered in calculus and its applications.

Not all functions are elementary. Non-elementary functions are also important with wide applications but they are outside our course of study.

Exponential function

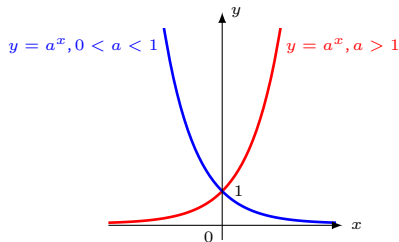
Let $a \in \mathbb{R}^+, a \neq 1$. The function $f : y = a^x$ is called **exponential function** with base a .

- ▶ The domain of f is \mathbb{R} and the range of f is $(0, \infty)$.
- ▶ If $a > 1$, f is increasing, if $0 < a < 1$ the function f is decreasing.
- ▶ The base $e \approx 2.71828182845 \dots$ is called **natural base**.
- ▶ For $a > 0$ and all $x, y \in \mathbb{R}$ the following holds

$$a^x \cdot a^y = a^{x+y}, \quad \frac{a^x}{a^y} = a^{x-y}, \quad (a^x)^y = a^{xy}.$$



The graphs of exponential functions are shown in the next figure.



Logarithmic function

The inverse function of the exponential function $y = a^x$ is called the **base a logarithmic function**. It is denoted $f : y = \log_a x$.

- ▶ The domain of f is $(0, \infty)$ and the range of f is \mathbb{R} .
- ▶ If $a > 1$, f is increasing, if $0 < a < 1$ the function f is decreasing.
- ▶ The base $e \approx 2.71828182845 \dots$ is called **natural base** and the natural logarithm is denoted $y = \ln x$. The base 10 is called **common base** and the common logarithm (or also the decadic logarithm) is denoted $y = \log x$.



- For every $a > 0$, $s \in \mathbb{R}$ and $x, y \in \mathbb{R}$ the following holds

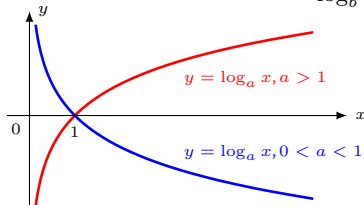
$$\log_a x + \log_a y = \log_a (x \cdot y)$$

$$\log_a x - \log_a y = \log_a \frac{x}{y}$$

$$\log_a x^s = s \log_a x.$$

- We can change the base of exponential or logarithm to another one.

$$a^x = e^{x \cdot \ln a}, \quad \log_a x = \frac{\log_b x}{\log_b a}.$$





Polynomials

Let $n \geq 1$ be a natural number, $a_0, a_1, \dots, a_{n-1}, a_n \in \mathbb{R}, a_n \neq 0$. A function of the form

$$P : y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, \quad x \in \mathbb{R},$$

is called a **polynomial function (polynomial)**, the elements $a_0, a_1, \dots, a_{n-1}, a_n$ are called **coefficients** of the polynomial, the integer n (the highest exponent on x) is the **degree** of the polynomial (denoted $\deg(P)$).

- ▶ The degree of the polynomial is the largest exponent k such that $a_k \neq 0$.
- ▶ The **zero polynomial** $P : y = 0$ has all the coefficients a_n equalled zero. This polynomial has no degree.
- ▶ The degree of $R : y = x^3$ is 3. The coefficients are $a_3 = 1, a_2 = a_1 = a_0 = 0$.
- ▶ The degree of $P : y = 3x^2 - 4x + 2$ is 2. The coefficients are $a_2 = 3, a_1 = -4, a_0 = 2$.
- ▶ The degree of $S : y = 2x - 3$ is 1. The coefficients are $a_1 = 2, a_0 = -3$.
- ▶ The degree of $T : y = 3$ is 0 and $a_0 = 3$.
- ▶ Polynomials of degree 0 are the **constant functions**, $f : y = a_0$. Their graphs are straight lines going through the point $(0, a_0)$.



- Polynomials of degree 1 are the **linear functions**, $f : y = a_1x + a_0$ where $a_1 \neq 0$. The coefficient a_1 is called the **slope** of the line and it determines its steepness. The constant a_0 is the y -intercept.

Any two points in the Cartesian plane determine a unique line. A line joining two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ has its slope

$$k = \frac{y_2 - y_1}{x_2 - x_1}.$$

The equation of the form $y = kx + q$ is called the **slope-intercept form** of the line with slope k and y -intercept $(0; q)$.

- Polynomials of degree 2 are the **quadratics functions**, $f : y = a_2x^2 + a_1x + a_0$ where $a_2 \neq 0$. We usually use notation $f(x) = ax^2 + bx + c$. The graph of a quadratic polynomial is a parabola. All of the graphs of quadratic functions can be created by transforming the parabola $y = x^2$.

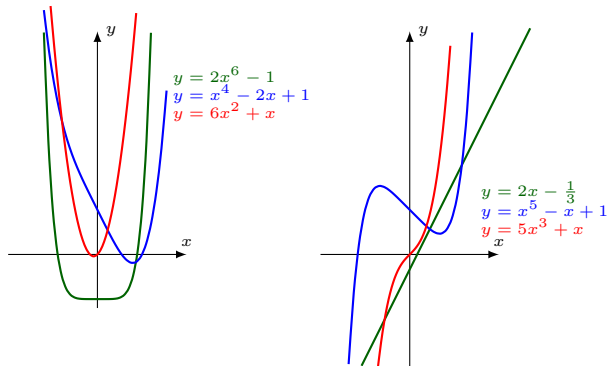
Every quadratic function can be put in the **standard form**

$$f(x) = a(x - x_0)^2 + y_0,$$

where a, x_0, y_0 are real numbers. The point (x_0, y_0) is the vertex of the graph $y = f(x)$.



Graphs of polynomials are smooth, with no holes nor sharp corners or cusps. The leading term $a_n x^n$ of a general polynomial is dominant over the other terms. It means that a polynomial of degree n has a shape very similar to $a_n x^n$ as the x -values approach the “ends” of the x -axis. Let us look at the following picture, where the graphs of polynomials of even degree and odd degree with a positive coefficient a_n are drawn.





Rational Function

A **rational function** is a function which is the ratio of polynomial functions. It means that R is a rational function if it is of the form

$$R(x) = \frac{P(x)}{Q(x)},$$

where P and Q are polynomial functions (Q is not the zero polynomial). The domain of a function R consists of all real numbers except the roots of Q . A rational function is called **proper** if $\deg(P) < \deg(Q)$.

- ▶ $f : y = \frac{4x^2 + 3x - 2}{x^5 - 3}$ – proper rational function.
- ▶ $g : y = \frac{3x^6 + 7}{2x^3 + 4x^2 + 1}$ – improper rational function.

If the degree of the denominator is less than the degree of the numerator, we can perform polynomial long division with remainder to decompose improper rational function R to the form

$$R(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{T(x)}{Q(x)},$$

where S, T are unique polynomials and $\deg(T) < \deg(Q)$.

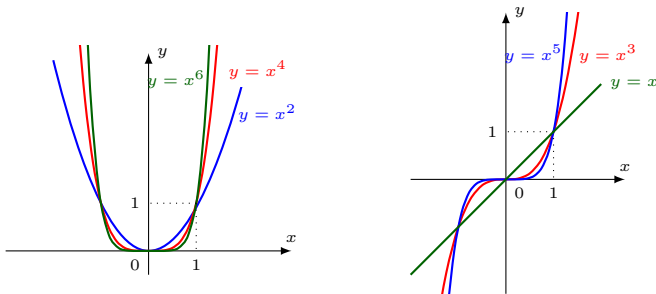


Power Function

A function $f : y = x^s$, where $s \in \mathbb{R}$, is called a **power function**. We will consider several different cases. For $s = n \in \mathbb{N}$ we define

$$f : y = x^n, x \in \mathbb{R}, \quad \text{where } x^n = \underbrace{x \cdot x \cdots x}_{n\text{--times}}.$$

The graphs of $f(x) = x^n$, for $n = 1, 2, 3, 4, 5, 6$, are displayed in the following figure.

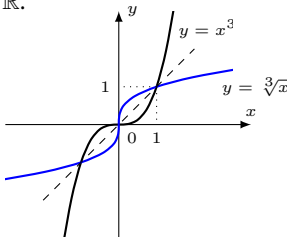
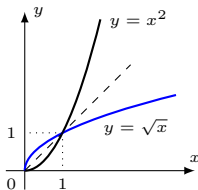




- The domain of f for $s = n \in \mathbb{N}$ is \mathbb{R} .
- For n even, the power x^n is an even function with $H_f = [0, \infty)$. It is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$.
- For n odd, the power x^n is an odd function with $H_f = \mathbb{R}$. It is increasing on \mathbb{R} .

For $s = \frac{1}{n}$, where $n \in \mathbb{N}$, we define the function $x^{1/n} = \sqrt[n]{x}$ as the **n -rooth** of x .

- For n even it is the inverse function of $f : y = x^n, x \in [0, \infty)$,
- For n odd it is the inverse function of $f : y = x^n, x \in \mathbb{R}$.

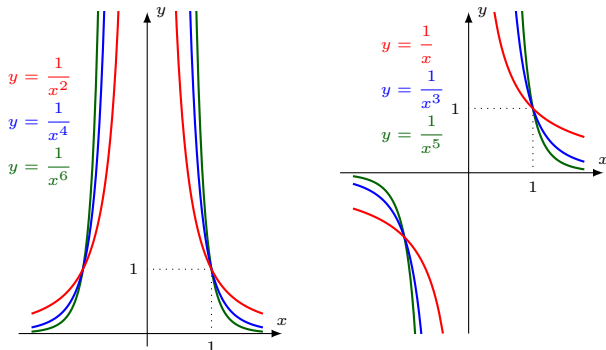




For $s = n \in \mathbb{N}$ we define

$$f : y = x^{-n}, x \in \mathbb{R} \setminus \{0\}, \quad \text{where } x^{-n} = \frac{1}{x^n} = \frac{1}{x \cdot x \cdots x},$$

The graphs of $f(x) = x^{-n}$, for $n = 1, 2, 3, 4, 5, 6$, are displayed in the following figure.



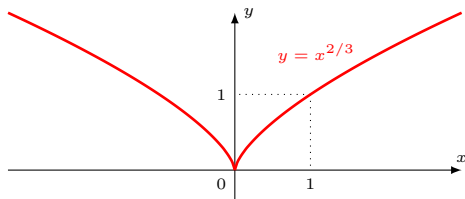


Let $s \in \mathbb{Q} - \mathbb{Z}$, $s = \frac{p}{q}$, where $p, q \in \mathbb{Z}$. Then

$$f(x) = x^{\frac{p}{q}} = (x^p)^{\frac{1}{q}} = \sqrt[q]{x^p}.$$

For $p/q > 0$ the function f is defined on $[0, \infty)$, for $p/q < 0$ the domain of f is $(0, \infty)$.

For $s \in \mathbb{R} - \mathbb{Q}$, so s irrational number, the domain of the function $f(x) = x^s$ is $(0, \infty)$.



For $x, y \in \mathbb{R}^+$, $a, b \in \mathbb{R}$, the following equalities hold

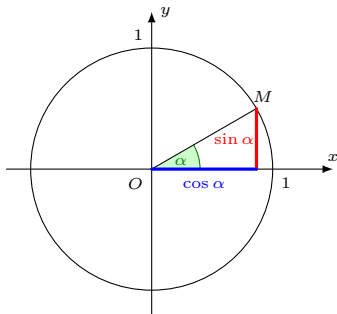
$$x^a x^b = x^{a+b}, \quad \frac{x^a}{x^b} = x^{a-b}, \quad x^{-a} = \frac{1}{x^a}, \quad (x^a)^b = x^{ab}.$$



Trigonometric functions

Trigonometric functions express relationships between angles and sides of a right-angled triangle. But they can be also defined using unit circle, which will be our case.

Let us consider a circle centered at the origin of the plain with a radius of 1 unit. Every point on the unit circle corresponds to a right triangle with vertices at the origin and the point on the unit circle.



This right triangle has a hypotenuse equaled to the radius 1 of the circle, an adjacent side is equaled to the x -coordinate of the point M and an opposite side equals to the y -coordinate.



We define the following trigonometric relations:

$$\sin \alpha = y, \quad \cos \alpha = x, \quad \tan \alpha = \frac{y}{x}, \quad \cot \alpha = \frac{x}{y}.$$

Functions $f(\alpha) = \sin \alpha$, $g(\alpha) = \cos \alpha$ represent x - and y -coordinates of the points on the unit circle, where α is measured in radians.

We usually measure the angles in degrees or radians. To measure in degrees, we consider 360 degrees to a full circle, it means a full revolution. By dividing the revolution into 360 parts we obtain each part of a size of 1° . The right angle is of a size of 90 degrees.

For a unit circle centred at the origin, the angle α measured in radians is the length of the arc carved out of the unit circle. For an arbitrary circle an angle size in radians is the ratio of the arc length to the radius.

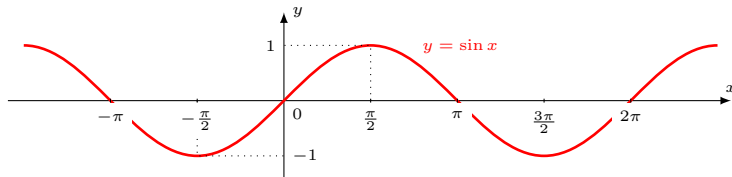
Since a full revolution is 360° or 2π radians, we have the following relationship between degrees and radians

$$1 \text{ radian} = \frac{180}{\pi} (\doteq 57.3) \text{ degrees}, \quad 1 \text{ degree} = \frac{\pi}{180} (\doteq 0.017) \text{ radians}.$$



Sine

The **sine** function $f : y = \sin x$ is defined for all $x \in \mathbb{R}$ and its range is $[-1, 1]$. This function is periodic with period $T = 2\pi$.



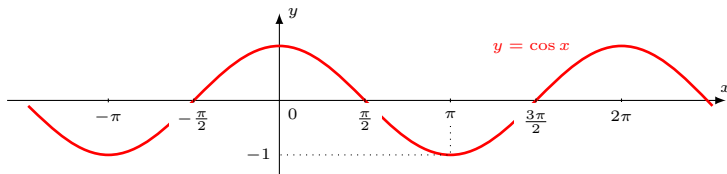
It is useful to remember the following values of the sine function.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1



Cosine

The **cosine** function $f : y = \cos x$ is defined for all $x \in \mathbb{R}$ and its range is $[-1, 1]$. This function is periodic with period $T = 2\pi$.



It is useful to remember the following values of the cosine function.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0



Tangent

The **tangent** function $f : y = \tan x = \frac{\sin x}{\cos x}$ has the domain

$$D(\tan) = \mathbb{R} - \left\{ \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \right\} = \bigcup_{k \in \mathbb{Z}} \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi \right).$$

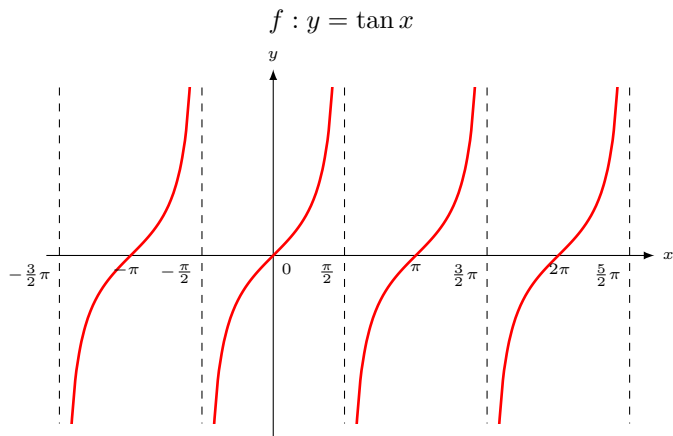
Cotangent

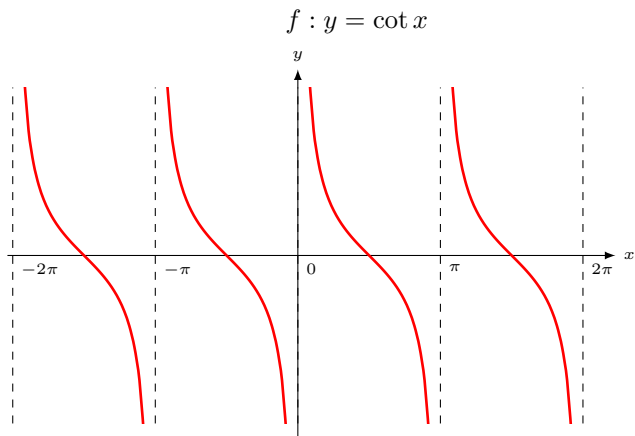
The **cotangent** function $f : y = \cot x = \frac{\cos x}{\sin x}$ has the domain

$$D(\cot) = \mathbb{R} - \{k\pi, k \in \mathbb{Z}\} = \bigcup_{k \in \mathbb{Z}} (k\pi, (k+1)\pi).$$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$
$\operatorname{tg} x$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	--	0	--
$\operatorname{cotg} x$	--	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	--	0

These functions are periodic with period $T = \pi$.







Trigonometric Identities

- ▶ $\sin \alpha = \frac{\text{opposite}}{\text{hypotenuse}}, \cos \alpha = \frac{\text{adjacent}}{\text{hypotenuse}}, \tan \alpha = \frac{\text{opposite}}{\text{adjacent}},$
- ▶ $\cos(-x) = \cos x, \sin(-x) = -\sin x, \tan(-x) = -\tan x, \cot(-x) = -\cot x,$
- ▶ $\sin^2 x + \cos^2 x = 1,$
- ▶ $\sin 2x = 2 \sin x \cos x,$
- ▶ $\cos 2x = \cos^2 x - \sin^2 x,$
- ▶ $\sin^2 x = \frac{1 - \cos 2x}{2},$
- ▶ $\cos^2 x = \frac{1 + \cos 2x}{2},$
- ▶ $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$
- ▶ $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$
- ▶ $\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2},$
- ▶ $\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2},$
- ▶ $\sin x \pm \sin y = 2 \sin \frac{x \pm y}{2} \cos \frac{x \mp y}{2}.$



Inverse Trigonometric Functions

The none of the four basic trigonometric functions does not have inverses. These functions are not one-to-one. But we can restrict their domains to suitable intervals to make each of them one-to-one. We will obtain graphs of inverse trigonometric functions by reflecting the graphs of the restricted trigonometric functions through the line $y = x$.

Arcsine

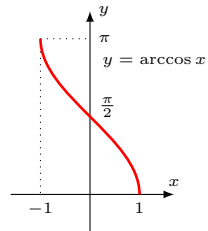
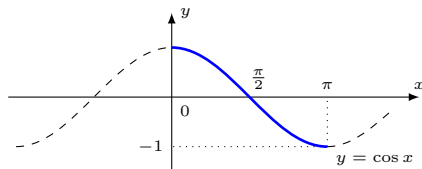
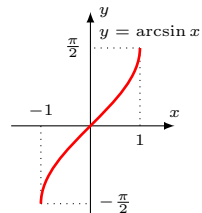
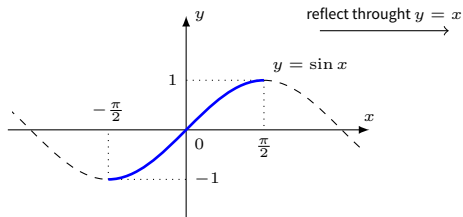
Let us consider the function $f : y = \sin x$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Its inverse function is called **arcsine** function. It is denoted $f^{-1} : y = \arcsin x$ (or $y = \sin^{-1} x$). The domain of arcsine is $[-1, 1]$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The arcsine function is odd. The relation between sine and arcsine functions is as follows:

$$\arcsin x = \alpha \quad \text{if and only if} \quad -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2} \quad \text{and} \quad \sin \alpha = x.$$

Arccosine

Let us consider the function $f : y = \cos x$ for $x \in [0, \pi]$. Its inverse function is called **arccosine** function. It is denoted $f^{-1} : y = \arccos x$ (or $y = \cos^{-1} x$). The domain of arccosine is $[-1, 1]$ and the range is $[0, \pi]$. The relation between cosine and arccosine functions is as follows:

$$\arccos x = \alpha \quad \text{if and only if} \quad 0 \leq \alpha \leq \pi \quad \text{and} \quad \cos \alpha = x.$$



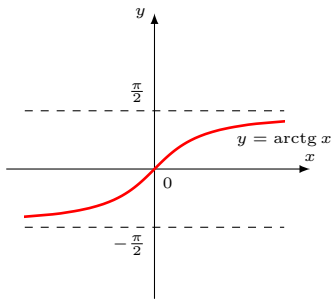
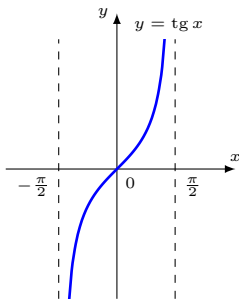


Arctangent

Let us consider the function $f : y = \tan x$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Its inverse function is called **arctangent** function. It is denoted $f^{-1} : y = \arctan x$ (or $y = \tan^{-1} x$). The domain of arctangent is \mathbb{R} and the range is $(-\frac{\pi}{2}, \frac{\pi}{2})$.

The arctangent function is odd. The relation between tangent and arctangent functions is as follows:

$$\arctan x = \alpha \quad \text{if and only if} \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2} \quad \text{and} \quad \tan \alpha = x.$$

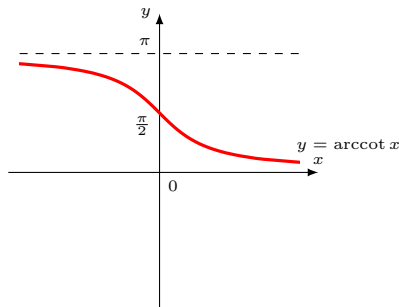
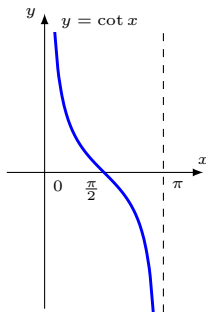




Arccotangent

Let us consider the function $f : y = \cot x$ for $x \in (0, \pi)$. Its inverse function is called **arccotangent** function. It is denoted $f^{-1} : y = \operatorname{arccot} x$ (or $y = \cot^{-1} x$). The domain of arccotangent is \mathbb{R} and the range is $(0, \pi)$. The relation between cotangent and arccotangent functions is as follows:

$\operatorname{arccot} x = \alpha$ if and only if $0 < \alpha < \pi$ and $\cot \alpha = x$.





Example 1.17

Evaluate

- a) $\arcsin \frac{1}{2}$, b) $\arcsin \left(-\frac{\sqrt{3}}{2}\right)$, c) $\arccos \left(-\frac{1}{2}\right)$, d) $\arctan 1$,
e) $\arctan (-\sqrt{3})$, f) $\operatorname{arccot} \frac{1}{\sqrt{3}}$.

Solution

- a) To find $\arcsin \frac{1}{2}$ we need to find the real number α (it means angle measured in radians) which lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\sin \alpha = \frac{1}{2}$. The number is $\alpha = \frac{\pi}{6}$.
- b) To find $\arcsin \left(-\frac{\sqrt{3}}{2}\right)$ we need to find the real number α which lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\sin \alpha = -\frac{\sqrt{3}}{2}$. The answer is $\alpha = -\frac{\pi}{3}$.
- c) The number $\alpha = \arccos \left(-\frac{1}{2}\right)$, which lies in $[0, \pi]$ and $\cos \alpha = -\frac{1}{2}$, is $\alpha = \frac{2}{3}\pi$.
- d) The number $\arctan 1$ is a number α between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\tan \alpha = 1$. We have $\alpha = \frac{\pi}{4}$.
- e) The number $\arctan (-\sqrt{3})$ is a number α in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with $\tan \alpha = (-\sqrt{3})$. We have $\alpha = -\frac{\pi}{3}$.
- f) The value $\operatorname{arccot} \frac{1}{\sqrt{3}}$ means that $\cot \alpha = \frac{1}{\sqrt{3}}$ for $\alpha \in (0, \pi)$. So, $\alpha = \frac{\pi}{3}$.



1.5 Transformations

We will present the changes of the graphs of basic elementary functions undergoing certain modifications.

Let f be a function and c be a positive number.

- ▶ The graph of $y = f(x) + c$ is the graph of the function f **shifted up** c units.
- ▶ The graph of $y = f(x) - c$ is the graph of the function f **shifted down** c units.
- ▶ The graph of $y = f(x + c)$ is the graph of the function f **shifted left** c units.
- ▶ The graph of $y = f(x - c)$ is the graph of the function f **shifted right** c units.
- ▶ The graph of $y = -f(x)$ is the graph of the function f **reflected across the x -axis**.
- ▶ The graph of $y = f(-x)$ is the graph of the function f **reflected across the y -axis**.
- ▶ The graph of $y = cf(x)$ for $c > 1$ is the graph of the function f **stretched (expanded) vertically** by a factor of c .
- ▶ The graph of $y = cf(x)$ for $0 < c < 1$ is the graph of the function f **compressed (shrunk) vertically** by a factor of $1/c$.
- ▶ The graph of $y = f(cx)$ for $c > 1$ is the graph of the function f **compressed (shrunk) horizontally** by a factor of c .
- ▶ The graph of $y = f(cx)$ for $0 < c < 1$ is the graph of the function f **stretched (expanded) horizontally** by a factor of $1/c$.



2. Limits and Continuity

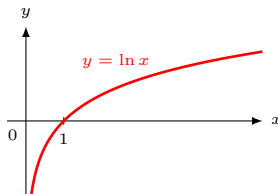
The concepts of limit and continuity are very important in the study of Calculus. Terms such as derivative of a function, Riemann integral and others in the next chapters are defined by limits.

The limit of a function enables us to investigate function behaviour in the neighbourhood of individual points, especially those, that do not belong to the domain. If we can not calculate the value of a function at x_0 , we count the limit of the given function for x “approaching” to x_0 . This process of “approaching” is described precisely by the term *Limit of a function*.



2.1 Motivation

Let us consider the function $y = \ln x$. Its domain is the interval $(0, \infty)$.



What happens to the curve if we go more left or right?

When we substitute increasing values of x , the function values are increasing indefinitely. We write

$$\lim_{x \rightarrow +\infty} \ln x = +\infty.$$

When we substitute decreasing values of x towards zero from the right, the function values are decreasing indefinitely. We write

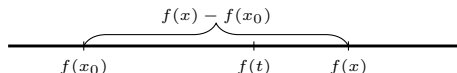
$$\lim_{x \rightarrow 0^+} \ln x = -\infty.$$



Here is the next example. Let us consider the ratio

$$\frac{f(x) - f(x_0)}{x - x_0}.$$

In physics, this ratio represents average speed of a body moving on the line, where $f(x)$ is its distance per the time x .

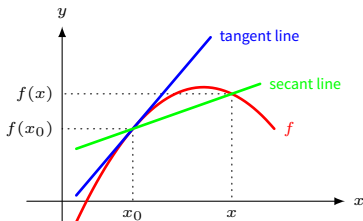


We are interested in the speed of an object at a single instant x_0 . It means what happens when the values x approach to x_0 (we write $x \rightarrow x_0$).

A geometric interpretation: the same ratio

$$\frac{f(x) - f(x_0)}{x - x_0}$$

represents the slope of the secant line to the graph of the function f , see next picture.



As we take x closer and closer to x_0 the slope of the secant line should be getting closer and closer to the slope of the tangent line. Then the limit

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

is called the **slope of the tangent line** to the graph of the function $f(x)$ at the point $(x_0, f(x_0))$.



2.2 Extended Real Line

In the next sections we will work with infinity, for instance, we will define a limit at infinity. We therefore define the **extended real line** as the real line plus two abstract elements $+\infty, -\infty$

$$\mathbb{R}^* = \mathbb{R} \cup \{\infty, -\infty\}.$$

The two points $\pm\infty$ are called **improper points**, the points from the set \mathbb{R} we call **proper points**. For $a \in \mathbb{R}$ it is defined

- ▶ $a + \infty = \infty, a - \infty = -\infty, \infty + \infty = \infty, -\infty - \infty = -\infty,$
- ▶ $\infty \cdot \infty = -\infty \cdot (-\infty) = \infty, \infty \cdot (-\infty) = -\infty, \frac{a}{\infty} = \frac{a}{-\infty} = 0, -\infty < a < \infty, |\pm\infty| = \infty,$
- ▶ for $a > 0$: $a \cdot \infty = \infty, a \cdot (-\infty) = -\infty$, for $a < 0$: $a \cdot \infty = -\infty, a \cdot (-\infty) = \infty$.

The operations “ $\infty - \infty$ ”, “ $\pm\infty \cdot 0$ ” a “ $\frac{\pm\infty}{\pm\infty}$ ” are undefined. The division by a zero “ $\frac{a}{0}$ ”, $a \in \mathbb{R}^*$, is undefined. Such expressions are called **indeterminate expressions**.

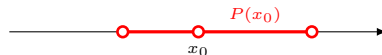
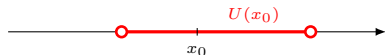


2.3 Neighborhood of a Point

An interval around the point will play an important role in defining the limit. We call such interval a **neighbourhood** of the point.

Definition 2.1 (Mařík, 2012)

Under the **neighbourhood** of the point $x_0 \in \mathbb{R}$ we understand any open interval which contains the point x_0 , we write $U(x_0)$. Under the **reduced neighborhood** of the point x_0 we understand the set $U(x_0) - \{x_0\}$, we write $P(x_0)$.



Definition 2.2 (Mařík, 2012)

Under the **neighbourhood of the point $+\infty$** we understand the interval of the type (K, ∞) , under the **neighbourhood of the point $-\infty$** we understand the interval of the type $(-\infty, L)$, where $K, L \in \mathbb{R}$.



2.4 The Definition of the Limit

Definition 2.3 (Mařík, 2012)

Let $x_0, L \in \mathbb{R}^*$. Let a function $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined in some reduced neighbourhood of the point x_0 . We say that the function $y = f(x)$ approaches to the **limit** L as x approaches to x_0 , if for any (arbitrary small) neighbourhood $U(L)$ of the number L there exists reduced neighbourhood $P(x_0)$ of the point x_0 such that for every $x \in P(x_0)$ the relation $f(x) \in U(L)$ holds. We write

$$\lim_{x \rightarrow x_0} f(x) = L$$

or $f(x) \rightarrow L$ for $x \rightarrow x_0$.

If a limit L is a real number, it is called a **proper limit**. A limit $+\infty$ or $-\infty$ is called **improper limit**. If we find any limit (proper or improper), we say that the limit **exists**.

The four possibilities for limit can arise. A limit at a proper point can be proper or improper and also a limit at an improper point can be a proper or an improper limit.

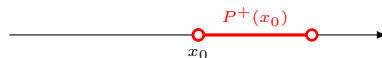
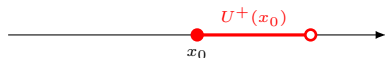


2.5 One-Sided Limits

A function f must be defined on both sides of x_0 to have a limit L as x tends to x_0 from either side. But a function f does not always have a limit at x_0 , or f can be not defined on both sides of the point x_0 but only to the left or right of x_0 . Thus we define the concept of **one-sided limit**, limit at x_0 from the left and limit at x_0 from the right.

Definition 2.4 (Mařík, 2012)

Under the **right (left) neighbourhood** of the point $x_0 \in \mathbb{R}$ we understand the interval of the type $[x_0, b)$, (or $(b, x_0]$ for left neighbourhood), we write $U^+(x_0)$ ($U^-(x_0)$). Under the **reduced right (left) neighbourhood** of the point x_0 we understand the corresponding neighbourhood without the point x_0 , we write $P^+(x_0)$ ($P^-(x_0)$).



A neighbourhood of the point ∞ is always left neighbourhood, a neighbourhood of the point $-\infty$ is always right neighbourhood. We do not talk about one-sided limits at improper points.



Definition 2.5 (Mařík, 2012)

If we replace in the definition of the limit the reduced neighbourhood of the point x_0 by the reduced right neighbourhood of the point x_0 , we obtain a definition of the **limit from the right**. We write $\lim_{x \rightarrow x_0^+} f(x) = L$.

We define the limit from the left similarly. In this case we write $\lim_{x \rightarrow x_0^-} f(x) = L$.

Theorem 2.6 (Bouchala & Sadowská, 2007)

A function f has at most one limit at $x_0 \in \mathbb{R}^$.*

Theorem 2.7 (Bouchala & Sadowská, 2007)

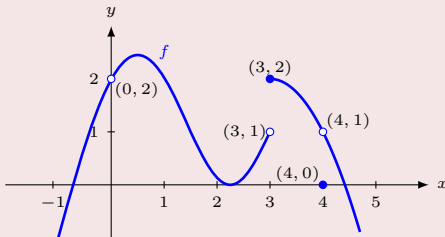
Let $x_0 \in \mathbb{R}$ and $L \in \mathbb{R}^$. Then $\lim_{x \rightarrow x_0} f(x) = L$ if and only if*

$$\lim_{x \rightarrow x_0^+} f(x) = \lim_{x \rightarrow x_0^-} f(x) = L.$$



Example 2.8

For the function f given by the graph



compute each of the following:

- | | | | |
|-----------|------------------------------------|------------------------------------|------------------------------------|
| a) $f(0)$ | b) $\lim_{x \rightarrow 0^-} f(x)$ | c) $\lim_{x \rightarrow 0^+} f(x)$ | d) $\lim_{x \rightarrow 0} f(x)$ |
| e) $f(3)$ | f) $\lim_{x \rightarrow 3^-} f(x)$ | g) $\lim_{x \rightarrow 3^+} f(x)$ | h) $\lim_{x \rightarrow 3} f(x)$ |
| i) $f(4)$ | j) $\lim_{x \rightarrow 4^-} f(x)$ | k) $\lim_{x \rightarrow 4^+} f(x)$ | l) $\lim_{x \rightarrow 4} f(x)$. |



Solution

- a) $f(0)$ does not exist.
- b) $\lim_{x \rightarrow 0^-} f(x) = 2.$
- c) $\lim_{x \rightarrow 0^+} f(x) = 2.$
- d) $\lim_{x \rightarrow 0} f(x) = 2$, the two one-sided limits are the same.
- e) $f(3) = 2$, the function will take on the y value where the closed dot is.
- f) $\lim_{x \rightarrow 3^-} f(x) = 1.$
- g) $\lim_{x \rightarrow 3^+} f(x) = 2.$
- h) $\lim_{x \rightarrow 3} f(x)$ does not exist.
- i) $f(4) = 0.$
- j) $\lim_{x \rightarrow 4^-} f(x) = 1.$
- k) $\lim_{x \rightarrow 4^+} f(x) = 1.$
- l) $\lim_{x \rightarrow 4} f(x) = 1.$



2.6 Limit Properties

We will show a few properties of limits. These properties will be useful in evaluating limits.

Theorem 2.9 (Hass, Giordano, Weir & Thomas, 2005)

Let $x_0 \in \mathbb{R}^*$, $A, B, c \in \mathbb{R}$ and $\lim_{x \rightarrow x_0} f(x) = A$, $\lim_{x \rightarrow x_0} g(x) = B$, then

- ▶ $\lim_{x \rightarrow x_0} [f(x) + g(x)] = \lim_{x \rightarrow x_0} f(x) + \lim_{x \rightarrow x_0} g(x) = A + B$ (the sum rule),
- ▶ $\lim_{x \rightarrow x_0} [f(x) - g(x)] = \lim_{x \rightarrow x_0} f(x) - \lim_{x \rightarrow x_0} g(x) = A - B$ (the difference rule),
- ▶ $\lim_{x \rightarrow x_0} [c \cdot f(x)] = c \cdot \lim_{x \rightarrow x_0} f(x) = c \cdot A$ (the constant multiple rule),
- ▶ $\lim_{x \rightarrow x_0} [f(x) \cdot g(x)] = \lim_{x \rightarrow x_0} f(x) \cdot \lim_{x \rightarrow x_0} g(x) = A \cdot B$ (the product rule),
- ▶ $\lim_{x \rightarrow x_0} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow x_0} f(x)}{\lim_{x \rightarrow x_0} g(x)} = \frac{A}{B}$ provided that $B \neq 0$ (the quotient rule).



In the theorem above we assumed that $A, B \in \mathbb{R}$, but some operations can be defined also for infinities. Indeterminate expressions $\infty - \infty$, $0 \cdot \infty$, ∞/∞ have to be handled individually. To compute the limit of more complex functions, we use the following theorem for the limit of a composite function.

Theorem 2.10 (Bouchala & Sadowská, 2007)

Let $y = f(g(x))$ be a composite function, $x_0 \in \mathbb{R}^*$, $A, B \in \mathbb{R}^*$. Further,

- i) let $\lim_{x \rightarrow x_0} g(x) = A$, $\lim_{u \rightarrow A} f(u) = B$,
- ii) let there exist reduced neighbourhood $P(x_0)$ of x_0 such that for all $x \in P(x_0)$ it holds $g(x) \neq A$ or f is continuous at A .

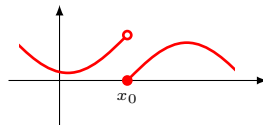
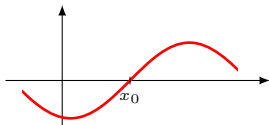
Then

$$\lim_{x \rightarrow x_0} f(g(x)) = B.$$



2.7 Continuity of the Function

The outcome of the limit does not depend on the value of the function f at a given point x_0 . The limit depends only on the function behaviour on immediate neighbourhood of the limit point. Look at the following graphs.



In both examples, the function has the same value at the point x_0 , so $f(x_0) = 0$. In the left picture the function looks “nicer”, because the limit at x_0 is exactly the same as the value of the function, you can draw the first graph without lifting your pencil off the paper and so on. We can say “what is happening around the point x_0 is exactly the same as what is happening at the point x_0 ”.

The function in the second example is more complicated. Its graph is interrupted at x_0 , the limit from the left at x_0 differs from the limit from the right. Furthermore, the limit from the right equals to the function value at the point x_0 .



Definition 2.11 (“Math Tutor”, 2019)

Let f be a function defined on some neighbourhood of a point x_0 . We say that the function f is **continuous** at x_0 iff

$$\lim_{x \rightarrow x_0} f(x) = f(x_0).$$

The function f is continuous at the point x_0 if

- a) $x_0 \in D_f$, it means $f(x_0)$ exists,
- b) $\lim_{x \rightarrow x_0} f(x)$ exists as a finite number,
- c) $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ holds.

Similarly, we define one-sided continuities. In the previous example, the second function is continuous from the right, because the curve goes straight to the point $(x_0, f(x_0))$.

Definition 2.12 (“Math Tutor”, 2019)

Let f be a function defined on some left neighbourhood of a point x_0 . We say that the function f is **continuous from the left** at x_0 iff $\lim_{x \rightarrow x_0^-} f(x) = f(x_0)$.



Definition 2.13 (“Math Tutor”, 2019)

Let f be a function defined on some right neighbourhood of a point x_0 . We say that the function f is **continuous from the right** at x_0 iff $\lim_{x \rightarrow x_0^+} f(x) = f(x_0)$.

From these definitions we can derive that a function f is continuous at some point x_0 if and only if it is continuous at x_0 both from the left and right.

Definition 2.14 (“Math Tutor”, 2019)

Let f be a function defined on some neighbourhood of a point x_0 . We say that x_0 is a point of **discontinuity** of f if f is not continuous at x_0 . We say that f is **discontinuous** at x_0 .

Definition 2.15 (“Math Tutor”, 2019)

The function is said to be **continuous on the open interval** (a, b) if it is continuous at every point of this interval. The function is said to be **continuous on the closed interval** $[a, b]$ if it is continuous on (a, b) , continuous from the right at the point a and continuous from the left at the point b .



Here are some important properties of continuous functions. It is easy to deduce them from the limit properties.

Theorem 2.16 (Mařík, 2012)

Every elementary function is continuous on its domain.

Theorem 2.17 (Bouchala & Sadowská, 2007)

Let functions f and g be continuous at $x_0 \in \mathbb{R}$. Then also functions $f + g$, $f - g$ and $f \cdot g$ are continuous at x_0 . If, moreover, $g(x_0) \neq 0$, then the function $\frac{f}{g}$ is continuous at x_0 .

Theorem 2.18 (Bouchala & Sadowská, 2007)

Let a function g be continuous at $x_0 \in \mathbb{R}$ and let a function f be continuous at $g(x_0)$. Then the function $y = f(g(x))$ is continuous at x_0 .

If f is continuous at b and $\lim_{x \rightarrow x_0} g(x) = b$, then

$$\lim_{x \rightarrow x_0} f(g(x)) = f\left(\lim_{x \rightarrow x_0} g(x)\right) = f(b).$$



The next **Extreme Value Theorem** (Weierstrass) guarantees the existence of both a maximum and minimum value for a continuous function on closed and bounded interval.

Theorem 2.19 (Weierstrass) (Mařík, 2012)

Let f be a function defined and continuous on $[a, b]$. Then the function f is bounded and takes on an absolute maximum and an absolute minimum on the interval $[a, b]$.

The consequences of Extreme Value Theorem are Rolle's theorem and the Mean Value Theorem, which are important in the study of differentiable functions. We will focus on differentiable functions in the next chapter.



2.8 Evaluating Limits

The class of continuous functions allows us to use $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ to compute limits. We get the limit if we evaluate the function f at the point x_0 .

Example 2.20

Evaluate $\lim_{x \rightarrow 2} (3x^2 - 8x)$.

Solution The function $y = 3x^2 - 8x$ is continuous for all $x \in \mathbb{R}$. Hence

$$\lim_{x \rightarrow 2} (3x^2 - 8x) = 3 \cdot 2^2 - 8 \cdot 2 = 12 - 16 = -4.$$

Example 2.21

Evaluate $\lim_{x \rightarrow 4} \ln \frac{x-3}{2x-7}$.

Solution We evaluate the limit of the composite function. We find the limit of the inside function first and then put this outcome into the outside function.



$$\lim_{x \rightarrow 4} \ln \frac{x-3}{2x-7} = \ln \lim_{x \rightarrow 4} \frac{x-3}{2x-7} = \ln \frac{\lim_{x \rightarrow 4} (x-3)}{\lim_{x \rightarrow 4} (2x-7)} = \ln \frac{1}{1} = \ln 1 = 0.$$

If we substitute x_0 into the f and it makes sense, then the value $f(x_0)$ is the limit of f at x_0 . If $f(x_0)$ is not defined, we must use another appropriate method. See solved problems.

Example 2.22

Evaluate $\lim_{x \rightarrow \infty} (3x^4 + 8x + 2)$.

Solution We try to plug infinity into the polynomial and evaluate each term. The limit of each term can be evaluated by an inspection of the graphs. As x approaches infinity, then x to a power approaches infinity, too.

$$\lim_{x \rightarrow \infty} (3x^4 + 8x + 2) = \infty + \infty + 2 = +\infty.$$

Example 2.23

Evaluate $\lim_{x \rightarrow -\infty} (3x^4 + 8x + 2)$.



Solution If we plug minus infinity into the polynomial we get the following

$$\lim_{x \rightarrow -\infty} (3x^4 + 8x + 2) = \infty - \infty + 2.$$

This is one of indeterminate forms. We need to factor the largest power of x out of the polynomial.

$$\begin{aligned}\lim_{x \rightarrow -\infty} (3x^4 + 8x + 2) &= \lim_{x \rightarrow -\infty} x^4 \cdot \left(3 + \frac{8}{x^3} + \frac{2}{x^4} \right) = \\ &= (-\infty)^4 \cdot \left(3 + \frac{8}{(-\infty)^3} + \frac{2}{(-\infty)^4} \right) = \infty \cdot \left(3 - \frac{8}{\infty} + \frac{2}{\infty} \right) = \\ &= \infty \cdot (3 - 0 + 0) = \infty.\end{aligned}$$

Theorem 2.24 (Mařík, 2012)

If $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_x + a_0$ is a polynomial of degree n (i.e. $a_n \neq 0$) then,

$$\lim_{x \rightarrow \pm\infty} p(x) = \lim_{x \rightarrow \pm\infty} a_n x^n.$$



Example 2.25

Evaluate $\lim_{x \rightarrow \infty} \frac{2x^3 - 4x}{1 - 3x^2}$.

Solution We factor out the largest power of x that is in the denominator from both the denominator and the numerator.

$$\lim_{x \rightarrow \infty} \frac{2x^3 - 4x}{1 - 3x^2} = \lim_{x \rightarrow \infty} \frac{x^2 \cdot (2x - \frac{4}{x})}{x^2 \cdot (\frac{1}{x^2} - 3)} = \lim_{x \rightarrow \infty} \frac{2x - \frac{4}{x}}{\frac{1}{x^2} - 3} = \frac{\infty}{-3} = -\infty.$$

Example 2.26

Evaluate $\lim_{x \rightarrow \infty} \frac{7x^4 - 2x^2 + x}{2x^4 + x - 2}$.

Solution We can use previous theorem and take the term with the largest power in the denominator and in the numerator and evaluate their limits.

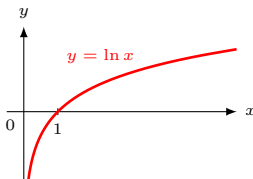
$$\lim_{x \rightarrow \infty} \frac{7x^4 - 2x^2 + x}{2x^4 + x - 2} = \lim_{x \rightarrow \infty} \frac{7x^4}{2x^4} = \frac{7}{2}.$$



Example 2.27

For $f(x) = \ln x$ evaluate $\lim_{x \rightarrow \infty} f(x)$, $\lim_{x \rightarrow 0^+} f(x)$.

Solution The easiest way to solve these problems is to investigate the graph of the natural logarithm.



As the argument of a logarithm approaches zero from the right, the function values will decrease indefinitely. Therefore $\lim_{x \rightarrow 0^+} \ln x = -\infty$.

As the argument of a logarithm increases, the function values will increase indefinitely, so $\lim_{x \rightarrow +\infty} \ln x = +\infty$.



Example 2.28

Evaluate $\lim_{x \rightarrow 1} \frac{x^2 + 2x - 3}{x^2 - x}$.

Solution If we plug $x = 1$ into the given function, we get

$$\lim_{x \rightarrow 1} \frac{x^2 + 2x - 3}{x^2 - x} = \frac{0}{0}.$$

It is indeterminate expression, but it does not mean that the limit does not exist. We need simplify the function as much as possible. Both the numerator and the denominator of a rational function are zero at $x = 1$, so the factor $(x - 1)$ can be factored and after that cancelled from both the numerator and the denominator. The limit is

$$\lim_{x \rightarrow 1} \frac{x^2 + 2x - 3}{x^2 - x} = \lim_{x \rightarrow 1} \frac{(x - 1)(x + 3)}{x(x - 1)} = \lim_{x \rightarrow 1} \frac{x + 3}{x} = \frac{4}{1} = 4.$$

The limits of the indeterminate type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ are usually solved by l'Hospital's Rule (see next chapter). But sometimes, cancelling is easier.



Now, let us assume that we want to find the limit of a function $f(x)/g(x)$ at some x_0 , where $\lim_{x \rightarrow x_0} f(x) = L \neq 0$ and $\lim_{x \rightarrow x_0} g(x) = 0$. This limit is the type $L/0$.

If there exists reduced neighbourhood $P(x_0)$ of the point x_0 so, that $\frac{f(x)}{g(x)} > 0$ for all $x \in P(x_0)$ then

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = +\infty.$$

If there exists reduced neighbourhood $P(x_0)$ of the point x_0 so, that $\frac{f(x)}{g(x)} < 0$ for all $x \in P(x_0)$ then

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = -\infty.$$

If the function $\frac{f(x)}{g(x)} > 0$ has different signs from the left side and from the right side of x_0 , then the limit of the type $L/0$ does not exist.

In the case of “ $L/0$ ” we often use one-sided limits and Theorem 2.7. If the two one-sided limits have different values then the normal limit will not exist.



Example 2.29

Evaluate $\lim_{x \rightarrow 2} \frac{x+1}{(x-2)^3}$.

Solution When we substitute the point $x = 2$ into $\frac{x+1}{(x-2)^3}$, we get division by zero, it means a limit of the type $L/0$.

We evaluate one-sided limits. If $x > 2$, then $\frac{x+1}{(x-2)^3} > 0$. The limit at 2 from the right is $+\infty$. If $x < 2$, then

$\frac{x+1}{(x-2)^3} < 0$. The limit at 2 from the left is $-\infty$. Since

$$\lim_{x \rightarrow 2^+} \frac{x+1}{(x-2)^3} \neq \lim_{x \rightarrow 2^-} \frac{x+1}{(x-2)^3},$$

the given limit does not exist.



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3. Differential Calculus

A major topic of Differential Calculus is the concept of Derivative. We will define the derivative of the function of one variable and will mention some of its interpretations. We will be looking at the most important applications of derivatives in this chapter.



3.1 Definition of the Derivative

Definition 3.1 (Bouchala & Sadowská, 2007)

For a function $f : \mathbb{R} \rightarrow \mathbb{R}$, its **derivative** at $x_0 \in \mathbb{R}$ is defined as

$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

if the limit exists. We read $f'(x_0)$ as “ f prime of x_0 ”.

With a small change of notation $h = x - x_0$, the limit can also be written as

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

A similar definition is used for one-sided derivatives from the right and from the left.

For given function $y = f(x)$ we can use an equivalent notation of its derivative

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}(f(x)).$$



Definition 3.2 (Dawkins, 2018)

A function f is called **differentiable at x_0** if $f'(x_0)$ exists. A function f is called **differentiable on an interval** if the derivative exists for each point in that interval.

Example 3.3

Find the derivative of the function $f(x) = x^2$ using the definition of the derivative.

Solution

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x \\ &\text{for all } x \in \mathbb{R}. \end{aligned}$$

For example, the value of the derivative f' at the point 3 is $f'(3) = 2 \cdot 3 = 6$.



Example 3.4

Find the derivative of the function $f(x) = |x|$ at 0.

Solution We have

$$\frac{f(0+h) - f(0)}{h} = \frac{|0+h| - |0|}{h} = \frac{|h|}{h}.$$

But

$$\lim_{h \rightarrow 0^-} \frac{|h|}{h} = -1, \quad \lim_{h \rightarrow 0^+} \frac{|h|}{h} = 1,$$

which implies that $f'(0)$ does not exist.

Theorem 3.5 (Bouchala & Sadowská, 2007)

Let f be a differentiable function at $x = x_0$. Then f is continuous at $x = x_0$.

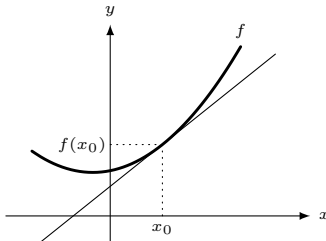
This theorem does not work in reverse. Function $f(x) = |x|$ is continuous at $x = 0$ (in \mathbb{R}), but is not differentiable at $x = 0$, see previous Example. A function need not have a derivative at a point where it is continuous.



Interpretations of the Derivative

- ▶ If $f(x)$ represents a quantity at any x , then the derivative $f'(x_0)$ represents the instantaneous rate of change of $f(x)$ at x_0 .
- ▶ The slope of the tangent line to $f(x)$ at x_0 is $f'(x_0)$. The tangent line is given by

$$y = f(x_0) + f'(x_0)(x - x_0).$$



- ▶ If the position of an object is given by $f(t)$ after t units of time, an instantaneous velocity of the object at $t = t_0$ is given by $f'(t_0)$.



3.2 Differentiation Formulas

It can be quite complicated to compute the derivative using the definition. We have a large collection of formulas and properties at our disposal to avoid using the definition whenever possible. The following formulas show how to compute a derivative of any basic elementary function.

$$\begin{aligned}(c)' &= 0, & x &\in (-\infty, \infty) \\(x^n)' &= nx^{n-1}, n > 0 & x &\in (-\infty, \infty) && \text{(the power rule)} \\(x^k)' &= kx^{k-1}, k < 0 & x &\in (-\infty, 0) \cup (0, \infty) \\(x^s)' &= sx^{s-1}, s \in \mathbb{R} & x &\in (0, \infty) \\(e^x)' &= e^x, & x &\in (-\infty, \infty) \\(a^x)' &= a^x \ln a, & x &\in (-\infty, \infty) \\(\ln x)' &= \frac{1}{x}, & x &\in (0, \infty) \\(\log_a x)' &= \frac{1}{x \ln a}, & x &\in (0, \infty) \\(\sin x)' &= \cos x, & x &\in (-\infty, \infty)\end{aligned}$$



$$\begin{aligned}(\cos x)' &= -\sin x, & x &\in (-\infty, \infty) \\(\operatorname{tg} x)' &= \frac{1}{\cos^2 x}, & x &\in \mathbb{R} - \bigcup_{k \in \mathbb{Z}} \left\{ (2k-1) \frac{\pi}{2} \right\} \\(\operatorname{cotg} x)' &= -\frac{1}{\sin^2 x}, & x &\in \mathbb{R} - \bigcup_{k \in \mathbb{Z}} \{k\pi\} \\(\arcsin x)' &= \frac{1}{\sqrt{1-x^2}}, & x &\in (-1, 1) \\(\arccos x)' &= -\frac{1}{\sqrt{1-x^2}}, & x &\in (-1, 1) \\(\operatorname{arctg} x)' &= \frac{1}{1+x^2}, & x &\in (-\infty, \infty) \\(\operatorname{arccotg} x)' &= -\frac{1}{1+x^2}, & x &\in (-\infty, \infty)\end{aligned}$$

Example 3.6

Differentiate the functions $f(x) = x^7$, $g(x) = \log_2 x$, $h(x) = 4^x$, $k(x) = \frac{1}{\sqrt{x}}$.

Solution

$$f'(x) = 7x^6, \quad g'(x) = \frac{1}{x \ln 2}, \quad h'(x) = 4^x \ln 4, \quad k'(x) = \left(x^{-\frac{1}{2}}\right)' = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2\sqrt{x^3}}.$$



3.3 Differentiation Rules

The following rules for differentiation allows to use previous formulas when evaluating derivatives of more complicated functions.

Theorem 3.7 (Mařík, 2012)

Let f, g be functions and $c \in \mathbb{R}$ be a real constant. The following relations hold

$$(c \cdot f(x))' = c \cdot f'(x) \quad \text{the constant multiple rule,}$$

$$(f(x) \pm g(x))' = f'(x) \pm g'(x) \quad \text{the sum rule,}$$

$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x) \quad \text{the product rule,}$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)} \quad \text{the quotient rule,}$$

whenever the derivatives on the right-hand side exist and the expression on the right-hand side is well defined.



Example 3.8

Differentiate the function $f(x) = 3x^6 + 2 \sin x$.

Solution We have the sum of two terms. We will differentiate each of the terms using the first and the second properties from above. For each term with a multiplicative constant we need to factor the constant out (using the constant multiple rule) and then do the derivative.

$$(3x^6 + 2 \sin x)' = (3x^6)' + (2 \sin x)' = 3(x^6)' + 2(\sin x)' = 3 \cdot 6 \cdot x^5 + 2 \cdot \cos x = 18x^5 + 2 \cos x.$$

Example 3.9

Differentiate the function $f(x) = \sqrt{x} \cdot e^x$.

Solution We will use the product rule. We need to first convert the square root to fractional exponent. We take the derivative of the first function times the second one and then add on to that the first function times the derivative of the second function,

$$(\sqrt{x} \cdot e^x)' = \left(x^{\frac{1}{2}}\right)' \cdot e^x + \sqrt{x} \cdot (e^x)' = \frac{1}{2} \cdot x^{-\frac{1}{2}} \cdot e^x + \sqrt{x} \cdot e^x = \frac{e^x}{2\sqrt{x}} + \sqrt{x} \cdot e^x.$$



Example 3.10

Differentiate the function $f(x) = \frac{x^3}{\ln x}$.

Solution

$$\left(\frac{x^3}{\ln x}\right)' = \frac{(x^3)' \cdot \ln x - x^3 \cdot (\ln x)'}{(\ln x)^2} = \frac{3x^2 \cdot \ln x - x^3 \cdot \frac{1}{x}}{\ln^2 x} = \frac{3x^2 \cdot \ln x - x^2}{\ln^2 x}.$$

Example 3.11

Differentiate the function $f(x) = \frac{xe^x}{x+2}$.

Solution We start with the quotient rule followed by the product rule.

$$\left(\frac{xe^x}{x+2}\right)' = \frac{(xe^x)' \cdot (x+2) - xe^x \cdot (x+2)'}{(x+2)^2} = \frac{(e^x + xe^x) \cdot (x+2) - xe^x \cdot 1}{(x+2)^2} = \frac{e^x (x^2 + 2x + 2)}{(x+2)^2}.$$



Theorem 3.12 (Chain Rule) (Mařík, 2012)

Let f and g be differentiable functions. The relation

$$[f(g(x))]' = f'(g(x))g'(x)$$

holds whenever the right hand side is well defined.

We use the Chain rule to differentiate a composite function. We first differentiate the “outside” function f leaving the inside function alone, then multiply the result by the derivative of the “inside” function $g(x)$.

Example 3.13

Differentiate the function $f(x) = \cos(2x^3 + x)$.

Solution The outside function is the cosine and the inside function is $2x^3 + x$.

$$f'(x) = \underbrace{-\sin}_{\text{derivative of outside function}} \underbrace{(2x^3 + x)}_{\text{leave inside function alone}} \cdot \underbrace{(6x^2 + 1)}_{\text{derivative of inside function}} = -(6x^2 + 1) \sin(2x^3 + x).$$



Example 3.14

Differentiate the function $f(x) = \left(\sqrt[3]{x} + \frac{2}{x}\right)^{10}$.

Solution The outside function is the exponent of 10 and the inside function is the function in parenthesis.

$$f'(x) = 10 \left(\sqrt[3]{x} + \frac{2}{x}\right)^9 \left(x^{\frac{1}{3}} + 2x^{-1}\right)' = 10 \left(\sqrt[3]{x} + \frac{2}{x}\right)^9 \cdot \left(\frac{1}{3}x^{-\frac{2}{3}} - 2x^{-2}\right).$$

We can rewrite the result into the form

$$f'(x) = 10 \left(\sqrt[3]{x} + \frac{2}{x}\right)^9 \cdot \left(\frac{1}{3\sqrt[3]{x^2}} - \frac{2}{x^2}\right).$$

Example 3.15

Differentiate the function $f(x) = \ln \sin 2x$.

Solution The function f is composed of three functions.



$$f'(x) = \frac{1}{\sin 2x} \cdot (\sin 2x)' = \frac{1}{\sin 2x} \cdot \cos 2x \cdot (2x)' = \frac{1}{\sin 2x} \cdot \cos 2x \cdot 2 = 2 \cotg 2x.$$

Example 3.16

Differentiate the function $f(x) = \sqrt{x \sin x}$.

Solution The outside function is the square root and the inside function is a product of the functions.

$$f'(x) = \left((x \sin x)^{\frac{1}{2}} \right)' = \frac{1}{2} (x \sin x)^{-\frac{1}{2}} (x \sin x)' = \frac{1}{2\sqrt{x \sin x}} (\sin x + x \cos x).$$



3.4 Higher Order Derivatives

Let f be a function and f' be the derivative of this function. If there exists derivative $(f')'$ of the function f' , then this derivative is said to be the second derivative of the function f and denoted f'' . If f'' is differentiable, its derivative, $(f'')' = f'''$, is the third derivative of f . For $n \in \mathbb{N}$ we obtain the n -th derivative $f^{(n)} = \left(f^{(n-1)}\right)'$ of the f .

Example 3.17

Find the first four derivatives for the function $f(x) = x^3 - \sin x + e^x$.

Solution

$$\begin{aligned}f'(x) &= 3x^2 - \cos x + e^x, & f''(x) &= 6x + \sin x + e^x, \\f'''(x) &= 6 + \cos x + e^x, & f^{(4)}(x) &= -\sin x + e^x.\end{aligned}$$

Example 3.18

Find the n -th derivative for the function $f(x) = \ln(x + 1)$.



Solution

$$f'(x) = \frac{1}{x+1}, \quad f''(x) = -\frac{1}{(x+1)^2}, \quad f'''(x) = \frac{1 \cdot 2}{(x+1)^3}, \dots,$$
$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(x+1)^n}.$$



3.5 Tangent and Normal lines

If the function f is differentiable at the point x_0 , then the line through the point $(x_0, f(x_0))$ with the slope $f'(x_0)$ is called **a tangent line to the graph of the function $f(x)$ at the point $(x_0, f(x_0))$** . The tangent line then is given by

$$t : y = f(x_0) + f'(x_0) \cdot (x - x_0). \quad (1)$$

The line perpendicular to the tangent line at the point of entry is called **a normal line to the graph of the function $f(x)$ at the point $(x_0, f(x_0))$** . The normal line then is given by

$$n : y = f(x_0) - \frac{1}{f'(x_0)}(x - x_0). \quad (2)$$

If $f'(x_0) = 0$, then the equation of the normal line n is $x = x_0$.

To find the equations of the tangent and normal lines we need to differentiate function f and replace x with x_0 in the $f(x)$ and $f'(x)$.



Example 3.19

Find the tangent and the normal line to the graph of the function $f(x) = x^3 - 2x$ at the point $T = (\frac{1}{2}, ?)$.

Solution We have the function $f(x) = x^3 - 2x$ and $x_0 = \frac{1}{2}$. A tangent line intersects the graph at the point $T = (x_0, f(x_0))$. We know only the first coordinate $x_0 = \frac{1}{2}$, so we need to find its the second coordinate. If we take the function value of the function f at $x_0 = \frac{1}{2}$, we get

$$f(x_0) = f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 2 \cdot \frac{1}{2} = -\frac{7}{8},$$

so $T = (\frac{1}{2}, -\frac{7}{8})$. Now, we find the derivative of the function f and its value at $x_0 = \frac{1}{2}$.

$$f'(x) = (x^3 - 2x)' = 3x^2 - 2 \quad \Rightarrow \quad f'\left(\frac{1}{2}\right) = 3 \cdot \left(\frac{1}{2}\right)^2 - 2 = -\frac{5}{4}.$$

The tangent and normal lines are

$$t : y = f(x_0) + f'(x_0) \cdot (x - x_0) \qquad n : y = f(x_0) - \frac{1}{f'(x_0)} \cdot (x - x_0)$$

$$y = -\frac{7}{8} - \frac{5}{4} \cdot \left(x - \frac{1}{2}\right)$$

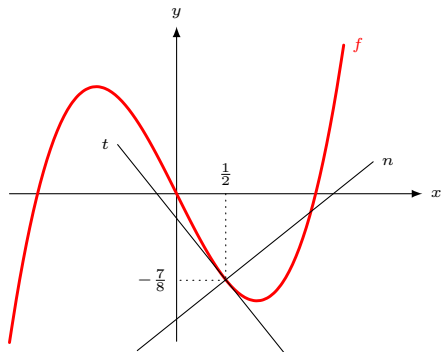
$$y = -\frac{7}{8} + \frac{4}{5} \cdot \left(x - \frac{1}{2}\right)$$

$$y = -\frac{5}{4}x - \frac{1}{4}$$

$$y = \frac{4}{5}x - \frac{51}{40}$$



Here are the graphs of the function f and its the tangent and normal line.

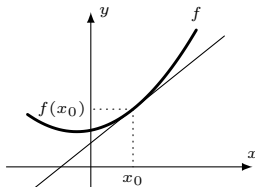




3.6 Linear Approximations

We will look at an application of the tangent line to the graph of a function. Let f be a function differentiable at the point $x = x_0$. The equation of the tangent line is

$$t(x) = f(x_0) + f'(x_0) \cdot (x - x_0).$$



From the graph we can see that the tangent is the best linear function which approximates $f(x)$ near the point x_0 . If x is really close to x_0 , then $f(x)$ is almost the same as $t(x)$, so we can approximate

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

We call the tangent line **the linear approximation** of the function at $x = x_0$.



Example 3.20

Approximate (without using calculator) the square root of 4.03.

Solution Let $f(x) = \sqrt{x}$. We want to know $f(4.03)$, so we need a point x_0 that is close to $x = 4.03$ and also is easy to put into the square root. It is obviously $x_0 = 4$. Now, we find the derivative of the function f and its value at $x_0 = 4$.

$$f'(x) = \left(x^{\frac{1}{2}}\right)' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \quad \Rightarrow \quad f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

Using above approximation, we get

$$\begin{aligned} f(x) &\approx f(x_0) + f'(x_0)(x - x_0) \\ f(4.03) &\approx f(4) + f'(4)(4.03 - 4) \\ \sqrt{4.03} &\approx 2 + \frac{1}{4} \cdot 0.03 \\ \sqrt{4.03} &\approx 2.0075 \end{aligned}$$

You can check our result with your calculator.



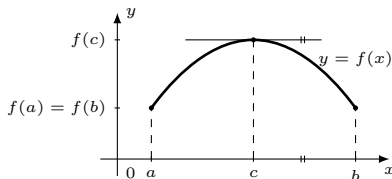
3.7 The Mean Value Theorem

We will introduce the most important theoretical tool in Calculus. The Mean Value Theorem allows us to proof further rules and theorems such as L'Hospital's Rule and others. At first we need to cover the following theorem.

Theorem 3.21 (Rolle's Theorem) (Herman, Strang et al, 2016a)

Let a function f be continuous on an interval $[a; b]$ and differentiable on $(a; b)$, and let $f(a) = f(b)$. Then there is a $c \in (a; b)$ such that $f'(c) = 0$.

A geometric interpretation of Rolle's Theorem: If the assumptions are satisfied, then there exists a point where the tangent line is horizontal.





Rolle's Theorem is as a special case of the Mean Value Theorem and it is needed in the proof of the Mean Value Theorem.

Theorem 3.22 (The Mean Value Theorem, Lagrange's theorem) (Hass, Giordano, Weir & Thomas, 2005)

Let a function $y = f(x)$ be continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b) . Then there is at least one point c in (a, b) at which

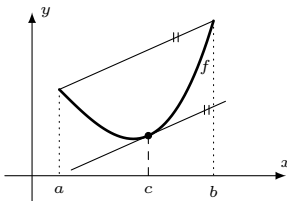
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

A geometric interpretation of the Mean Value Theorem: There is a point where the tangent line is parallel to the secant line connecting the two endpoints of the graph of f . Their slopes are equal and are exactly the fraction on the right.

A physical interpretation of the Mean Value Theorem: If the ratio on the right is the average velocity over the given interval and the expression on the left is the instantaneous velocity at a certain time, then at some interior point the instantaneous change must equal the average change over the entire interval.



Geometric interpretation of the Mean Value Theorem



The following theorems are the consequences of the Mean Value Theorem.

Theorem 3.23 (Dawkins, 2018)

If $f'(x) > 0$ ($f'(x) < 0$) for all x in an interval (a, b) , then f increasing (decreasing) on (a, b) .

Theorem 3.24 (Mařík, 2012)

If $f'(x) = g'(x)$ for all $x \in (a, b)$, then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$.



A generalization of the Mean Value Theorem is the following theorem.

Theorem 3.25 (Cauchy's Mean Value Theorem) (Bouchala & Sadowská, 2007)

Let functions f and g be continuous on an interval $[a; b]$ and differentiable on $(a; b)$. Let g' be finite and nonzero on $(a; b)$. Then there is a $c \in (a; b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Cauchy's theorem is the most powerful of the three above mentioned theorems dealing with the mean value. L'Hospital's rule in the next section can be proved by Cauchy's Mean Value Theorem.



3.8 L'Hospital's Rule

When evaluating limits of fractions, we often get the ratios like $0/0$ or ∞/∞ , which are called **indeterminate forms**.

Theorem 3.26 (l'Hospital's Rule) (Mařík, 2012)

Let $x_0 \in \mathbb{R}^*$ and let f and g be functions defined and differentiable in some ring neighbourhood of the point x_0 . Suppose that either

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = 0 \quad \text{or} \quad \left| \lim_{x \rightarrow x_0} g(x) \right| = \infty$$

holds. Then

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$$

holds if the limit on the right-hand side exists (finite or infinite). The same holds for one-sided limits as well.

If the right limit exists, then the left one exists and these two limits are equal, otherwise the equality need not be true. L'Hospital's Rule works on the two indeterminate forms $0/0$ and ∞/∞ .



We must differentiate the numerator and differentiate the denominator and then evaluate the limit. If this new limit is again one of the indeterminate forms, we can repeat l'Hospital's rule.

Example 3.27

Evaluate the limit $\lim_{x \rightarrow \infty} \frac{\ln x}{x^3}$.

Solution We will check the assumptions of the l'Hospital rule, it means to evaluate the limits of the numerator and denominator

$$\lim_{x \rightarrow \infty} \ln x = \infty, \quad \lim_{x \rightarrow \infty} x^3 = \infty.$$

Assigned limit is a ∞/∞ indeterminate form. We will apply l'Hospital's Rule to f/g , it means to divide the derivative of f by the derivative of g .

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x^3} = \left[\frac{\infty}{\infty} \right] = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{3x^2} = \lim_{x \rightarrow \infty} \frac{1}{3x^3} = 0.$$



Example 3.28

Evaluate the limit $\lim_{x \rightarrow 1} \frac{x^3 - 6x^2 + 3x + 2}{4x^4 - 3x^2 - 1}$.

Solution In this case we have a 0/0 indeterminate form. We can apply l'Hospital's Rule.

$$\lim_{x \rightarrow 1} \frac{x^3 - 6x^2 + 3x + 2}{4x^4 - 3x^2 - 1} = \left[\frac{0}{0} \right] = \lim_{x \rightarrow 1} \frac{3x^2 - 12x + 3}{16x^3 - 6x} = \lim_{x \rightarrow \infty} \frac{3 - 12 + 3}{16 - 6} = -\frac{3}{5}.$$

Example 3.29

Evaluate the limit $\lim_{x \rightarrow 0^+} \frac{\ln(e^x - 1)}{\ln x}$.

Solution Assigned limit is a ∞/∞ indeterminate form.

$$\lim_{x \rightarrow 0^+} \frac{\ln(e^x - 1)}{\ln x} = \lim_{x \rightarrow 0^+} \frac{\frac{e^x}{e^x - 1}}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{xe^x}{e^x - 1}.$$

This new limit is a 0/0 indeterminate form. We can continue to differentiate.



$$\lim_{x \rightarrow 0^+} \frac{\ln(e^x - 1)}{\ln x} = \lim_{x \rightarrow 0^+} \frac{xe^x}{e^x - 1} = \lim_{x \rightarrow 0^+} \frac{e^x + xe^x}{e^x} = \frac{1 + 0}{1} = 1.$$

It was necessary to use l'Hospital's Rule more than once.

.....
There are other types of indeterminate forms such as

$$[0 \cdot \infty], [\infty - \infty], [0^0], [\infty^0], [1^\infty].$$

We can convert these forms to a $0/0$ or ∞/∞ form and try to use l'Hospital's Rule to them.

- If the limit is a $[0 \cdot \infty]$ form, we write a product of functions as a quotient by writing one of the functions as a fraction:

$$\lim_{x \rightarrow x_0} f(x)g(x) = \lim_{x \rightarrow x_0} \frac{f(x)}{\frac{1}{g(x)}}, \quad \lim_{x \rightarrow x_0} f(x)g(x) = \lim_{x \rightarrow x_0} \frac{g(x)}{\frac{1}{f(x)}}.$$

These limits are a $0/0$ or ∞/∞ forms.



- If the limit is a $[\infty - \infty]$ form, we can use the following

$$\lim_{x \rightarrow x_0} [f(x) - g(x)] = \lim_{x \rightarrow x_0} \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)g(x)}}.$$

We can try algebraic simplification into one fraction (in the form of some common denominator) if both terms are fractions. In some special cases limits may be found by factoring out, cancellation or other algebraic manipulations.

- If the limit is one of the forms $[0^0]$, $[\infty^0]$, $[1^\infty]$, we can use the following

$$\lim_{x \rightarrow x_0} f(x)^{g(x)} = \lim_{x \rightarrow x_0} e^{g(x) \ln f(x)} = e^{\lim_{x \rightarrow x_0} g(x) \ln f(x)}.$$



Example 3.30

Evaluate the limit $\lim_{x \rightarrow 0^+} x \ln^2 x$.

Solution Assigned limit is a $0 \cdot \infty$ indeterminate form. Let's transform the product into a fraction.

$$\begin{aligned}\lim_{x \rightarrow 0^+} x \ln^2 x &= \lim_{x \rightarrow 0^+} \frac{\ln^2 x}{\frac{1}{x}} = \left[\frac{\infty}{\infty} \right] \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} \frac{2 \ln x \cdot \frac{1}{x}}{-\frac{1}{x^2}} = -2 \cdot \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} = \\ &\stackrel{\text{L'H}}{=} -2 \cdot \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 2 \cdot \lim_{x \rightarrow 0^+} x = 0.\end{aligned}$$

It is important to rightly change a product into a fraction, because the transformation

$$x \cdot \ln^2 x = \frac{x}{\frac{1}{\ln^2 x}}$$

does not lead to a result, although the limit of this function for $x \rightarrow 0^+$ is a $0/0$ form.



Example 3.31

Evaluate the limit $\lim_{x \rightarrow 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x} \right)$.

Solution Assigned limit is a $\infty - \infty$ indeterminate form. Let's transform the difference of fractions in the form of common denominator.

$$\begin{aligned} \lim_{x \rightarrow 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x} \right) &= \lim_{x \rightarrow 1^+} \frac{\ln x - x + 1}{(x-1) \ln x} = \left[\frac{0}{0} \right] \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 1^+} \frac{\frac{1}{x} - 1}{\ln x + \frac{x-1}{x}} = \\ &= \lim_{x \rightarrow 1^+} \frac{\frac{1-x}{x}}{\frac{x \ln x + x - 1}{x}} = \lim_{x \rightarrow 1^+} \frac{1-x}{x \ln x + x - 1} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} \frac{-1}{\ln x + \frac{x}{x} + 1} = \\ &= \frac{-1}{0 + 1 + 1} = -\frac{1}{2}. \end{aligned}$$



Example 3.32

Evaluate the limit $\lim_{x \rightarrow 0} (\cos 3x)^{\frac{1}{x^2}}$.

Solution This is the indeterminate form 1^∞ . At first let's rewrite the given expression

$$(\cos 3x)^{\frac{1}{x^2}} = e^{\frac{1}{x^2} \cdot \ln(\cos 3x)} = e^{\frac{\ln(\cos 3x)}{x^2}}.$$

Let's use L'Hospital's Rule on the power.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\ln(\cos 3x)}{x^2} &= \left[\frac{0}{0} \right] \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{\frac{-3 \sin 3x}{\cos 3x}}{2x} = -\frac{3}{2} \cdot \lim_{x \rightarrow 0} \frac{\tan 3x}{x} = \\ &\stackrel{\text{L'H}}{=} -\frac{3}{2} \cdot \lim_{x \rightarrow 0} \frac{\frac{3}{\cos^2 3x}}{1} = -\frac{3}{2} \cdot 3 = -\frac{9}{2}. \end{aligned}$$

We can now finish our problem.

$$\lim_{x \rightarrow 0} (\cos 3x)^{\frac{1}{x^2}} = e^{\lim_{x \rightarrow 0} \frac{\ln(\cos 3x)}{x^2}} = e^{-\frac{9}{2}}.$$



3.9 Monotonicity of the Function

We will show how to find the intervals of monotonicity of a function using its derivative. From the Mean Value Theorem we can deduce the following theorem.

Theorem 3.33 (Bouchala & Sadowská, 2007)

Let a function f be continuous on an interval I with end points $a, b \in \mathbb{R}^$, $a < b$.*

- ▶ *If $f'(x) > 0$ for all $x \in (a, b)$, then the function f is increasing on the interval I .*
- ▶ *If $f'(x) \geq 0$ for all $x \in (a, b)$, then the function f is non-decreasing on the interval I .*
- ▶ *If $f'(x) < 0$ for all $x \in (a, b)$, then the function f is decreasing on the interval I .*
- ▶ *If $f'(x) \leq 0$ for all $x \in (a, b)$, then the function f is non-increasing on the interval I .*
- ▶ *If $f'(x) = 0$ for all $x \in (a, b)$, then the function f is constant on the interval I .*

This theorem does not work in reverse. For example the function $f(x) = x^3$ is increasing at $x_0 = 0$, but its derivative is $f'(0) = 0$.



Example 3.34

Determine all intervals where the function $f(x) = x^3 - 12x$ is increasing or decreasing.

Solution Function f is defined for $x \in (-\infty, \infty)$. We find the derivative

$$f'(x) = 3x^2 - 12 = 3 \cdot (x^2 - 4) = 3(x + 2)(x - 2), \quad x \in \mathbb{R}.$$

We need to determine where the derivative is positive and where it is negative. The derivative has two zero points

$$3(x + 2)(x - 2) = 0 \quad \Rightarrow \quad x_1 = -2, \quad x_2 = 2.$$

These points divide the number line into three intervals

$$(-\infty, -2), \quad (-2, 2), \quad (2, \infty).$$

We will pick test points from each region to see if the derivative is positive or negative in each region.

$$x \in (-\infty, -2): \quad f'(x) > 0,$$

$$x \in (-2, 2): \quad f'(x) < 0,$$

$$x \in (2, \infty): \quad f'(x) > 0.$$



It follows from the Theorem 3.33 that

- ▶ f is increasing on $(-\infty, -2]$ and on $[2, \infty)$,
- ▶ f is decreasing on $[-2, 2]$.

Example 3.35

Determine all intervals where the function $f(x) = \frac{x^2}{x-1}$ is increasing or decreasing.

Solution Function f is defined for $x \in (-\infty, 1) \cup (1, \infty)$. We find the derivative

$$f'(x) = \frac{2x(x-1) - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}, \quad x \neq 1.$$

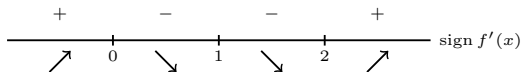
We solve the equation $f'(x) = 0$.

$$\frac{x(x-2)}{(x-1)^2} = 0 \quad \Rightarrow \quad x(x-2) = 0 \quad \Rightarrow \quad x_1 = 0, x_2 = 2.$$

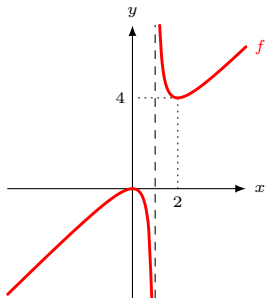
The derivative is not defined for $x_0 = 1$. This point and the points $x_1 = 0, x_2 = 2$ divide the real line into three intervals $(-\infty, 0), (0, 1), (1, 2), (2, \infty)$ on which f' is either positive or negative.



We determine the sign of f' by evaluating f' at a convenient point in each interval.



The function f is increasing on $(-\infty, 0]$ and on $[2, \infty)$ and decreasing on $[0, 1]$ and on $(1, 2]$.





3.10 Local Extrema

Besides monotonicity we are interested in minimum and maximum values of a function. In practical situations we are looking for optimal solutions like to maximize profit or minimize costs. Problems of this type are called problems on optimization.

Definition 3.36 (Bouchala & Sadowská, 2007)

Let $x_0 \in D_f$. A function f has

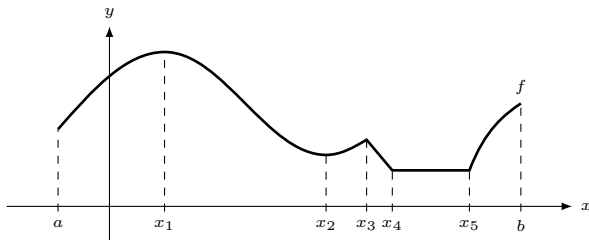
- ▶ a **local maximum** at x_0 if there is a neighbourhood $P(x_0)$ such that

$$\forall x \in P(x_0) : f(x) \leq f(x_0);$$

- ▶ a **local minimum** at x_0 if there is a neighbourhood $P(x_0)$ such that

$$\forall x \in P(x_0) : f(x) \geq f(x_0).$$

The name **local extrema** (or relative extrema) is a general designation for a local maximum or a local minimum. Sometimes we distinguish **sharp local extreme** (or strict one) when the inequalities are taken sharp.



The function f shown in this graph has the following properties.

- ▶ There are sharp local maximums at $x = x_1, x = x_3$.
- ▶ There is a sharp local minimum at $x = x_2$.
- ▶ There are local minimums and local maximums at the same $x \in (x_4, x_5)$.

Theorem 3.37 (Necessary Condition for Existence of a Local Extreme) (Bouchala & Sadowská, 2007)

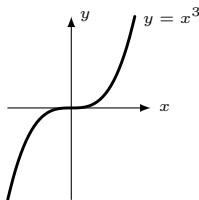
If a function f has a local extreme at $x_0 \in \mathbb{R}$, then either $f'(x_0) = 0$ or $f'(x_0)$ does not exist.



This statement is not true in reverse. The condition $f'(x_0) = 0$ does not imply the existence of a local extremum at the point x_0 .

Example 3.38

The function $f(x) = x^3$ for $x_0 = 0$ has $f'(0) = 0$, but there is no local extrema of the function f at $x_0 = 0$.



Definition 3.39 (“S.O.S. Mathematics”, 2019)

A point x_0 is said to be a **stationary point** of the function f if $f'(x_0) = 0$. Stationary points and points, where f' does not exist, are called **critical points**.



Theorem 3.40 (Fermat's Theorem) (Dawkins, 2018)

If f has a local extreme at $x = x_0$ and $f'(x_0)$ exists, then $x = x_0$ is a stationary point of f and thus $f'(x_0) = 0$.

To find local extrema we first find all critical points and then investigate them. For classifying extrema we have the following theorem which shows relationship between critical point and monotonicity of the function.

Theorem 3.41 (Sufficient Conditions for (Non-)Existence of Local Extrema) (Mařík, 2012)

Let f be continuous function at x_0 .

- ▶ *If there is a right neighbourhood of x_0 on which f is increasing and a left neighbourhood of x_0 on which f is decreasing, then f has a local minimum at x_0 .*
- ▶ *If there is a right neighbourhood of x_0 on which f is decreasing and a left neighbourhood of x_0 on which f is increasing, then f has a local maximum at x_0 .*



If the function f is elementary, we can find local extrema and intervals of monotonicity in the following steps.

1. Find intervals of the domain of f .
2. Find the derivative f' . Find critical points, it means points, where $f'(x) = 0$ or f' does not exist.
3. Mark points from Step 2 on the real line. These points divide the real line into several subintervals. Determine the sign of the derivative on each subinterval. We substitute convenient points from each subinterval to f' to find the sign.
4. Determine monotonicity from the signs of f' . The function f is increasing on intervals where f' is positive. It is decreasing on intervals where f' is negative.
5. Determine local extrema. If f changes from decreasing to increasing and is continuous at the point x_0 , then there is a local minimum at the point x_0 . If f changes from increasing to decreasing and is continuous at the point x_0 , then there is a local maximum at the point x_0 .



Example 3.42

Find local extrema and intervals of monotonicity of the function $f(x) = xe^{-2x}$.

Solution Function f is defined for $x \in (-\infty, \infty)$. We find the derivative

$$f'(x) = e^{-2x} + xe^{-2x} \cdot (-2) = e^{-2x}(1 - 2x), \quad x \in \mathbb{R}.$$

There are no points where f' does not exist. We solve the equation $f'(x) = 0$ to find critical points.

$$e^{-2x}(1 - 2x) = 0 \quad \Rightarrow \quad 1 - 2x = 0 \quad \Rightarrow \quad x_0 = \frac{1}{2}.$$

This point divides the number line into two intervals

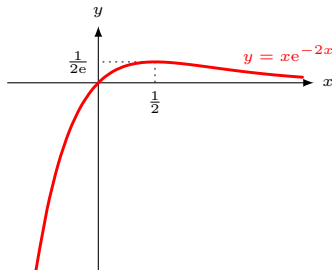
$$\left(-\infty, \frac{1}{2}\right), \quad \left(\frac{1}{2}, \infty\right).$$

Let's consider test points $c_1 = 0, c_2 = 1$ from these intervals. The derivative at these points is $f'(0) = 1$ and $f'(1) = -e^{-2}$. We obtain the following scheme.

$$\begin{array}{c} + \qquad \text{MAX} \qquad - \\ \hline \nearrow \quad \frac{1}{2} \quad \searrow \end{array} \quad \text{sign } f'(x)$$



Function f is increasing on $(-\infty, \frac{1}{2}]$ and decreasing on $[\frac{1}{2}, \infty)$. Critical point $x_0 = \frac{1}{2}$ was candidate for a local extreme. The value $f(\frac{1}{2}) = \frac{1}{2e}$ is a local maximum.





Example 3.43

Find local extrema and intervals of monotonicity of the function $f(x) = \frac{x^3}{x-2}$.

Solution Function f is defined for $x \in (-\infty, 2) \cup (2, \infty)$. We find the derivative

$$f'(x) = \frac{3x^2(x-2) - x^3}{(x-2)^2} = \frac{2x^3 - 6x^2}{(x-2)^2} = \frac{2x^2(x-3)}{(x-2)^2}, \quad x \neq 2.$$

The derivative f' does not exist at $x_0 = 2$, this point does not belong to D_f . We solve the equation $f'(x) = 0$ to find critical points.

$$\frac{2x^2(x-3)}{(x-2)^2} = 0 \quad \Rightarrow \quad 2x^2(x-3) = 0 \quad \Rightarrow \quad x_1 = 0, x_2 = 3.$$

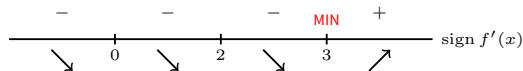
The derivative is not defined for $x_0 = 2$. This point and the points $x_1 = 0$, $x_2 = 3$ divide the real line into four intervals

$$(-\infty, 0), \quad (0, 2), \quad (2, 3), \quad (3, \infty).$$

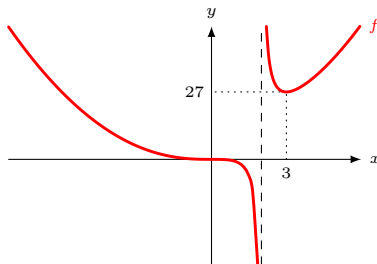
Let's consider test points $c_1 = -1$, $c_2 = 1$, $c_3 = \frac{5}{2}$, $c_4 = 4$ from these intervals.



We get $f'(-1) < 0$, $f'(1) < 0$, $f'(\frac{5}{2}) < 0$, $f'(4) > 0$. We draw the following scheme.



Function f is increasing on $[3, \infty)$ and decreasing on $(-\infty, 2)$, $(2, 3]$. Critical points $x_1 = 0$, $x_2 = 3$ were candidates for local extrema. There is no local extreme at $x = 0$, the value $f(3) = 27$ is a local minimum.





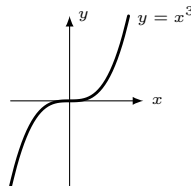
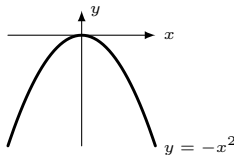
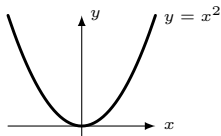
We can use the second and higher order derivatives to classify some of the critical points of a function.

Theorem 3.44 (Second Derivative Test, Sufficient Condition for Existence of a Local Extreme) (Mařík, 2012)

Let x_0 be a stationary point of a function f ($f'(x_0) = 0$).

- ▶ If $f''(x_0) > 0$, then the function f has its local minimum at the point x_0 .
- ▶ If $f''(x_0) < 0$, then the function f has its local maximum at the point x_0 .

If $f''(x_0) = 0$, the second derivative test fails. There can be a local maximum, a local minimum or neither at the point x_0 . All three possibilities are sketched at the following graphs. All of three chosen function have a critical point at $x = 0$, the second derivative of all of the functions is zero at $x = 0$.





The first is the graph of $f(x) = x^2$. This graph has a relative minimum at $x = 0$. The next is the graph of $f(x) = -x^2$. This graph has a relative maximum at $x = 0$. The latest graph shows the function $f(x) = x^3$. This graph has neither a relative minimum or a relative maximum at $x = 0$.

An advanced version of the previous theorem removes problems if $f''(x) = 0$.

Theorem 3.45 (“Math Tutor”, 2019)

Let $x_0 \in D_f$. Let n be the smallest integer for which

$$f'(x_0) = f''(x_0) = \dots = f^{(n-1)}(x_0) = 0 \quad \wedge \quad f^{(n)}(x_0) \neq 0.$$

- ▶ If n is odd, the function f does not have a local extreme at the point x_0 .
- ▶ If n is even and $f^{(n)}(x_0) > 0$, the function f has a strict local minimum at the point x_0 .
- ▶ If n is even and $f^{(n)}(x_0) < 0$, the function f has a strict local maximum at the point x_0 .

There can still be points where the first derivative does not exist. By this reason we usually do the classification using intervals of monotonicity.



Example 3.46

Find local extrema of the function $f(x) = 2x^4 + 3x^3 - 1$.

Solution Function f is defined and differentiable for all $x \in \mathbb{R}$. So, it can have local extrema only at stationary points. We find the derivative

$$f'(x) = 8x^3 + 9x^2 = x^2(8x + 9), \quad x \in \mathbb{R}.$$

We solve the equation $f'(x) = 0$ to find critical points.

$$x^2(8x + 9) = 0 \quad \Rightarrow \quad x_1 = 0, x_2 = -\frac{9}{8}.$$

The second derivative is

$$f''(x) = 24x^2 + 18x.$$

The values of the second derivative for each of critical points are

$$f''(0) = 0, \quad f''\left(-\frac{9}{8}\right) = \frac{81}{8}.$$



The second derivative at $x_2 = -\frac{9}{8}$ is positive and so we have a relative minimum here. At the point $x_1 = 0$ the second derivative is zero, we need to determine the next derivative at this point.

$$f'''(x) = 48x + 18 \quad \Rightarrow \quad f'''(0) = 18 \neq 0.$$

The number 3 is odd, the function f does not have a local extreme at the point $x_1 = 0$. We can find that the function f is increasing at this point.



3.11 Global Extrema

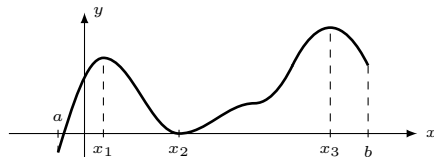
We will show other type of minimum or maximum values.

Definition 3.47 (Dawkins, 2018)

Let M be a non-empty subset of the domain of a function f .

- ▶ We say that f has an **absolute (or global) maximum** at $x_0 \in M$, if $f(x_0) \geq f(x)$ for every x in the set M .
- ▶ We say that f has an **absolute (or global) minimum** at $x_0 \in M$, if $f(x_0) \leq f(x)$ for every x in the set M .

An absolute maximum (or minimum) of a function f is the largest (or smallest) value that the function will ever take on the domain that we are working on.





The function f shown in the previous graph has the following properties.

- ▶ There are local maximums at $x = x_1, x = x_3$.
- ▶ There is a local minimum at $x = x_2$.
- ▶ There is a global minimum at $x = a$.
- ▶ There is a global maximum at $x = x_3$.

It is generally true that a function does not have to have any kind of extrema, local or absolute. But if a function is continuous at every point of a **closed** interval $[a, b]$, it has an absolute maximum and an absolute minimum value on this interval.

Theorem 3.48 (Extreme Value Theorem, Weierstrass Theorem) (Hass, Giordano, Weir & Thomas, 2005)

Let f be a function defined and continuous on $[a, b]$. Then the function f is bounded and takes on an absolute maximum and an absolute minimum on the interval $[a, b]$, i.e. there exist numbers $x_1, x_2 \in [a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a, b]$.

If the interval is not bounded or closed or the function is not continuous on the interval, then there is no guarantee that a f will have global extrema.



Theorem 3.49 (“Math Tutor”, 2019)

Let f be a defined on a bounded closed interval I . If f attains its global extreme at a point x_0 , then either f has a local extreme at x_0 or x_0 is an endpoint of I .

We can find global extrema in the following steps.

1. Verify that the function is continuous on the interval $[a, b]$.
2. Find the critical points of f that are in the interval $[a, b]$.
3. Evaluate the function f at the calculated critical points and the endpoints a, b .
4. Choose a global maximum and minimum from among these values. Global maximum is the largest value that the function attains and global minimum is the smallest value.

Example 3.50

Find global extrema of the function $f(x) = x^4 - 8x^2 + 1$ for $x \in [-3, 1]$.

Solution We will follow the procedure given above. The function f is a polynomial, so it is continuous for all $x \in \mathbb{R}$ and in particular is continuous on the given interval. We differentiate the function f to find its critical points.

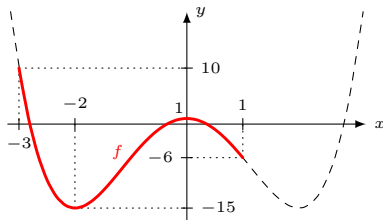


$$f'(x) = 4x^3 - 16x = 4x \cdot (x^2 - 4) = 0 \Rightarrow x_1 = -2, x_2 = 0, x_3 = 2.$$

We only want the critical points of the function f that lie in the interval in question. Therefore, we only want $x_1 = -2, x_2 = 0$, because $x_3 = 2$ falls outside the interval $[-3, 1]$. Now we evaluate the function at the critical points and the endpoints of the interval.

$$f(-2) = -15, \quad f(0) = 1, \quad f(-3) = 10, \quad f(1) = -6.$$

Absolute extrema are the largest and smallest values of the function. The absolute maximum of the f is 10 and it occurs at $x = -3$ (an endpoint). The absolute minimum of the f is -15 and it occurs at $x = -2$ (a critical point).





3.12 Concavity

In the previous section we used the first derivative to get intervals, where a function is increasing, where it is decreasing and to determine local extrema. The second derivative gives us further information about the graph of a differentiable function.

Definition 3.51 (Mařík, 2012)

Let f be a function differentiable at x_0 . We say that f is **concave up** (**concave down**) at x_0 if there is a neighborhood U of x_0 such that for all $x \in U$ the points on the graph of the function f are above (below) the tangent to the graph at the point x_0 , i.e. if

$$f(x) > f(x_0) + f'(x_0)(x - x_0) \quad (f(x) < f(x_0) + f'(x_0)(x - x_0))$$

holds.

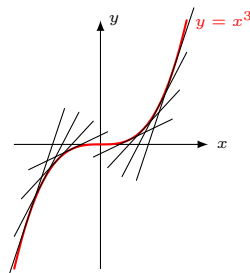
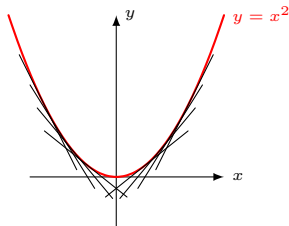
A function is concave up on an open interval exactly, if it is concave up at all points of it, an analogous statement is true about concavity down.

We can also say convex when we mean concave up, and concave when we mean concave down.



Definition 3.52 (Mařík, 2012)

The point x_0 in which the type of concavity changes is said to be a **point of inflection** of the function f .



The curve $f(x) = x^2$ is concave up on the interval $(-\infty, \infty)$.

The function $f(x) = x^3$ is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, \infty)$. The graph of this function changes from being concave down to concave up at the point $x = 0$. This point is called **inflection point**.



As you can see in the two previous graphs, if the function f is concave up, then the rate of the increase of the function f is increasing (the slopes of the tangents are increasing), if the function f is concave down, then the rate of the increase slows down. We deduce that f' increases if $f'' > 0$, and decreases if $f'' < 0$.

Theorem 3.53 (Bouchala & Sadowská, 2007)

Let f be a continuous function in an interval I and $f''(x)$ exists at every interior point x of I . Then f is

- ▶ *convave up in I if and only if $f''(x) \geq 0$ at every interior point x of I ;*
- ▶ *concave down in I if and only if $f''(x) \leq 0$ at every interior point x of I .*

At a point of inflection x_0 , either the second derivative exists and $f''(x_0) = 0$ or $f''(x_0)$ does not exist.

Example 3.54

Investigate concavity and inflection points of the function $f(x) = 2x^4 - 3x^3$.

Solution The function f is continuous and differentiable for all $x \in \mathbb{R}$. We have

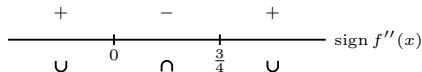


$$f'(x) = 8x^3 - 9x^2, \quad f''(x) = 24x^2 - 18x.$$

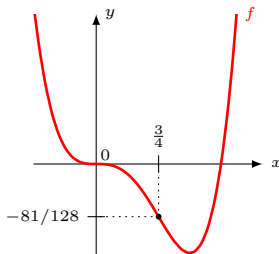
We solve the equation $f''(x) = 0$ to find the possible inflection points.

$$24x^2 - 18x = 0 \quad \Rightarrow \quad 6x \cdot (4x - 3) \quad \Rightarrow \quad x_1 = 0, x_2 = \frac{3}{4}.$$

Just as we identify the intervals of monotonicity we can draw a number line up and use these points to divide the number line into regions. In these regions the second derivative will always have the same sign since these two points are the only places where the second derivative may change sign. We need to pick a point from each region and plug it into the second derivative. The second derivative will then have that sign in the whole region from which the point came from.



We have got the following intervals of concavity, the function f is concave down on $(0, \frac{3}{4})$, it is concave up on $(-\infty, 0)$ and $(\frac{3}{4}, \infty]$. The function f has two points of inflection $x_1 = 0, x_2 = \frac{3}{4}$.



Example 3.55

Investigate concavity and inflection points of the function $f(x) = x^{\frac{13}{7}} - x$.

Solution The function f is continuous for all $x \in \mathbb{R}$. We have

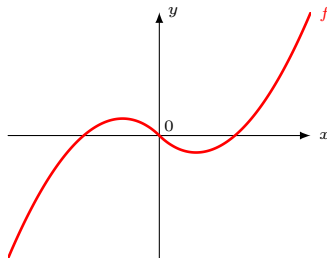
$$f'(x) = \frac{13}{7}x^{\frac{6}{7}} - 1 \quad \text{for all } x \in \mathbb{R}$$



and

$$f''(x) = \frac{78}{49}x^{-\frac{1}{7}} \quad \text{for all } x \neq 0.$$

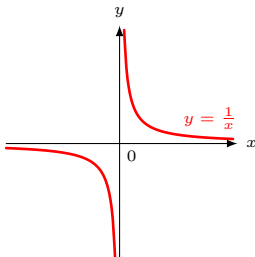
The second derivative of the function f at 0 does not exist. We can find that $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so the second derivative changes sign at $x = 0$. There is a point of inflection at the origin, even though $f''(0)$ does not exist.





3.13 Asymptotes

Let us look at the graph of the function $f(x) = \frac{1}{x}$ for $x \neq 0$.



If a point on the graph of this function moves increasingly far from the origin, then the distance between the graph and the x -axis approaches zero. We write

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0, \quad \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

and we say that the line $y = 0$ is a **horizontal asymptote** of the graph of $f(x) = \frac{1}{x}$.

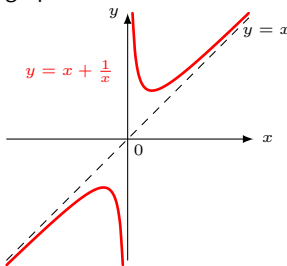


If a point on the graph of the function $f(x) = \frac{1}{x}$ moves vertically far from the origin, then the distance between the graph and the y -axis approaches zero. We write

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty, \quad \lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$$

and we say that the line $x = 0$ is a **vertical asymptote** of the graph of $f(x) = \frac{1}{x}$.

The last graph, the graph of the function $f(x) = x + \frac{1}{x}$, shows another kind of asymptotes, so called a **oblique asymptote**. The graph of the f tends to the graph of the line as x tends to infinity (or minus infinity).



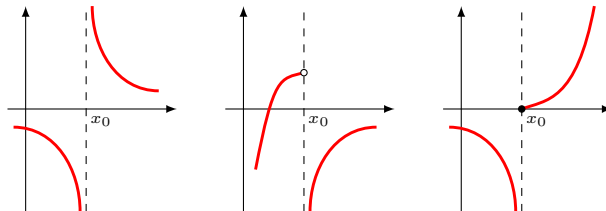


Definition 3.56 (Hass, Giordano, Weir & Thomas, 2005)

A line $x = x_0$ ($x_0 \in \mathbb{R}$) is called a **vertical asymptote** of the graph of a function f if at least one of the one-sided limits of the function f at the point x_0 is improper, i.e. if either

$$\lim_{x \rightarrow x_0^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \rightarrow x_0^+} f(x) = \pm\infty.$$

There are four limits in the definition but only one of the limits is sufficient for a function to have a vertical asymptote at x_0 . If f is continuous at x_0 , then it cannot have a vertical asymptote there. So only proper endpoints of intervals forming the domain and the points inside these intervals where f may not be continuous are candidates for vertical asymptotes.



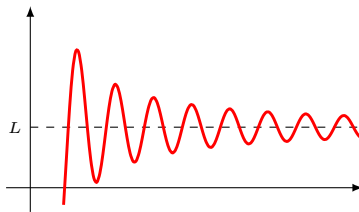
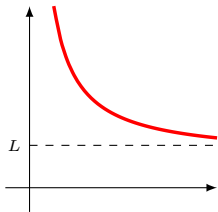


Definition 3.57 (Hass, Giordano, Weir & Thomas, 2005)

A line $y = L$ ($L \in \mathbb{R}$) is called a **horizontal asymptote** of the graph of a function f if either

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow +\infty} f(x) = L.$$

If the limit of a function f at infinity (or minus infinity) is proper and equal to L , then the line $y = L$ is the horizontal asymptote of f at infinity (or minus infinity). Horizontal asymptotes belong among oblique ones which we will define in the next definition.





Definition 3.58 (Hass, Giordano, Weir & Thomas, 2005)

A line $y = kx + q$ ($k, q \in \mathbb{R}$) is called

- ▶ an **oblique asymptote** of the graph of a function f at infinity if

$$\lim_{x \rightarrow +\infty} [f(x) - (kx + q)] = 0.$$

- ▶ an **oblique asymptote** of the graph of a function f at minus infinity if

$$\lim_{x \rightarrow -\infty} [f(x) - (kx + q)] = 0.$$

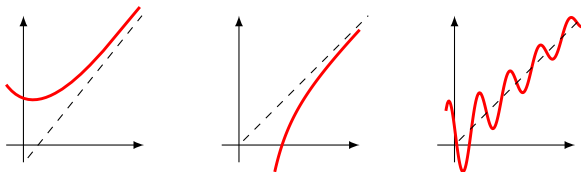
These asymptotes are also called inclined asymptotes or slant line asymptotes.

There exists no relationship between the asymptote at $-\infty$ and ∞ . We must find both asymptotes separately.

The horizontal asymptote is a special case of an oblique asymptote at infinity or negative infinity for $k = 0$.



The graph of a function can tend to its oblique asymptote from above or from below or it can move from side to side of an asymptote.



The following theorem gives the instructions how to find the asymptotes of a given function f at $+\infty$ and at $-\infty$.

Theorem 3.59 (Bouchala & Sadowská, 2007)

A line $y = kx + q$ is an oblique asymptote of f at infinity if and only if

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = k, \quad \lim_{x \rightarrow +\infty} [f(x) - kx] = q.$$

Analogous statement is true for asymptote at minus infinity.



Example 3.60

Find all asymptotes of the graph of the function $f(x) = \frac{3x^2 - 8x - 1}{4x - 12}$.

Solution Function f is defined for $x \in (-\infty, 3) \cup (3, \infty)$. The point $x = 3$ is the proper endpoint of intervals of the domain, so it is a candidate for a vertical asymptote. Since

$$\lim_{x \rightarrow 3^+} \frac{3x^2 - 8x - 1}{4x - 12} = \left[\frac{2}{0^+} \right] = +\infty,$$

the function f has a vertical asymptote $x = 3$. We do not have to evaluate the second one-sided limit, we already know from this limit that f has a vertical asymptote at 3. The limit

$$\lim_{x \rightarrow \infty} \frac{3x^2 - 8x - 1}{4x - 12} = \lim_{x \rightarrow \infty} \frac{3x^2}{4x} = \infty$$

shows that there is no horizontal asymptote at infinity.



The limit

$$\lim_{x \rightarrow -\infty} \frac{3x^2 - 8x - 1}{4x - 12} = \lim_{x \rightarrow -\infty} \frac{3x^2}{4x} = -\infty$$

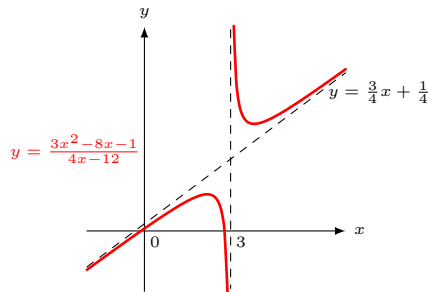
shows that there is no horizontal asymptote at minus infinity. Now, we test the existence of the oblique asymptotes.

$$k = \lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \frac{\frac{3x^2 - 8x - 1}{4x - 12}}{x} = \lim_{x \rightarrow +\infty} \frac{3x^2 - 8x - 1}{4x^2 - 12x} = \lim_{x \rightarrow +\infty} \frac{3x^2}{4x^2} = \frac{3}{4},$$

$$\begin{aligned} q &= \lim_{x \rightarrow +\infty} [f(x) - kx] = \lim_{x \rightarrow +\infty} \left[\frac{3x^2 - 8x - 1}{4x - 12} - \frac{3}{4}x \right] = \\ &= \lim_{x \rightarrow +\infty} \frac{3x^2 - 8x - 1 - 3x^2 + 9x}{4x - 12} = \lim_{x \rightarrow +\infty} \frac{x - 1}{4x - 12} = \frac{1}{4}. \end{aligned}$$

The same computations can be extended also to the case of limits at $-\infty$. We conclude that the line $y = \frac{3}{4}x + \frac{1}{4}$ is the asymptote at infinity and also minus infinity of the graph of f .

The graph of f is shown on the next slide.



Example 3.61

Find all asymptotes of the graph of the function $f(x) = \frac{\ln(x-1)}{x-2}$.

Solution We need to avoid logarithm of negative numbers and division by zero. Therefore, the function f is defined for $x \in (1, 2) \cup (2, \infty)$. The points $x = 1, x = 2$ are the endpoints of intervals of the domain.



Since

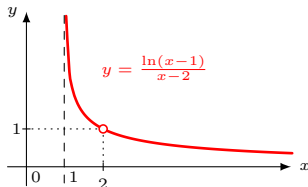
$$\lim_{x \rightarrow 1^+} \frac{\ln(x-1)}{x-2} = \left[\frac{-\infty}{-1} \right] = +\infty,$$

$$\lim_{x \rightarrow 2^-} \frac{\ln(x-1)}{x-2} = \left[\frac{0}{0} \right] \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 2^-} \frac{\frac{1}{x-1}}{1} = 1, \quad \lim_{x \rightarrow 2^+} \frac{\ln(x-1)}{x-2} = \left[\frac{0}{0} \right] \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 2^+} \frac{\frac{1}{x-1}}{1} = 1,$$

the function f has a vertical asymptote $x = 1$. The limit

$$\lim_{x \rightarrow \infty} \frac{\ln(x-1)}{x-2} = \left[\frac{\infty}{\infty} \right] \stackrel{\text{L'H}}{=} \lim_{x \rightarrow +\infty} \frac{\frac{1}{x-1}}{1} = 0$$

shows that there is a horizontal asymptote at infinity $y = 0$.





3.14 Investigation of a Function Behaviour

We will summarize the main points from this chapter. We need to determine characteristic properties and detailed information about the function to sketch its graph. To investigate the function behaviour means to follow this procedure.

1. Determine the domain of f , symmetry and periodicity.
2. Find intercepts of the graph with axes and intervals where the value of the function is positive and/or negative.
3. Find the one-sided limits at the points of discontinuity and at $\pm\infty$.
4. Find all asymptotes.
5. Find the derivative f' . Determine intervals of monotonicity and local extrema.
6. Find the second derivative f'' . Determine intervals of concavity and inflection points.
7. Draw the graph.

There are two possibilities. We can get all the information first then we do the sketch. Or at each step we sketch acquired data into the picture.



Example 3.62

Investigate the behaviour of the function $f(x) = x^3 + 4x^2 + 4x$.

Solution Function f is defined for all $x \in (-\infty, \infty)$. Since

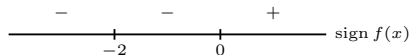
$$f(-x) = (-x)^3 + 4(-x)^2 + 4(-x) = -x^3 + 4x^2 - 4x$$

the function f is not even, nor odd. We find the intercepts of the graph with axes and intervals on which the graph is below and above the x -axis.

$$x^3 + 4x^2 + 4x = 0 \quad \Leftrightarrow \quad x(x+2)^2 = 0 \quad \Leftrightarrow \quad x = -2 \vee x = 0$$

$$x = 0 \Rightarrow f(0) = 0$$

The intercepts $x = -2, x = 0$ divide the real line into three intervals and we have



The function f is negative for all $x < 0, x \neq -2$, and positive for all $x > 0$. We find limits at the endpoints of the interval $(-\infty, \infty)$ given by the domain.



$$\lim_{x \rightarrow \pm\infty} (x^3 + 4x^2 + 4x) = \lim_{x \rightarrow \pm\infty} x^3 \cdot \left(1 + \frac{4}{x} + \frac{4}{x^2}\right) = \pm\infty$$

Since the function is continuous on the whole real line, there cannot be vertical asymptotes. Because limits at negative infinity and infinity diverge, there are no horizontal asymptotes there. We test the existence of the oblique asymptotes.

$$k_1 = \lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} (x^2 + 4x + 4) = +\infty$$

$$k_2 = \lim_{x \rightarrow -\infty} \frac{f(x)}{x} = \lim_{x \rightarrow -\infty} (x^2 + 4x + 4) = +\infty$$

The limits for k_1, k_2 diverged, therefore there are no oblique asymptotes there. We find the derivative

$$f'(x) = 3x^2 + 8x + 4, \quad x \in \mathbb{R}.$$

We need to determine where the derivative is positive and where it is negative. The derivative has two zero points

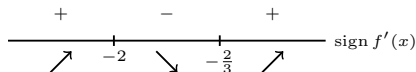
$$3x^2 + 8x + 4 = 0 \quad \Rightarrow \quad x_{1,2} = \frac{-8 \pm \sqrt{64 - 4 \cdot 3 \cdot 4}}{6} \quad \Rightarrow \quad x_1 = -2, \quad x_2 = -\frac{2}{3}.$$



These points divide the number line into three intervals

$$(-\infty, -2), \quad (-2, -\frac{2}{3}), \quad (-\frac{2}{3}, \infty).$$

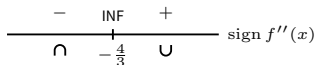
We will pick test points from each region to see if the derivative is positive or negative in each region. We determine the sign of f' by evaluating f' at a convenient point in each interval.



The function f is increasing on $(-\infty, -2]$ and on $[-\frac{2}{3}, \infty)$ and decreasing on $[-2, -\frac{2}{3}]$. The value $f(-2) = 0$ is a local maximum and the value $f(-\frac{2}{3}) = -\frac{32}{27}$ is a local minimum. We find the second derivative and solve the equation $f''(x) = 0$.

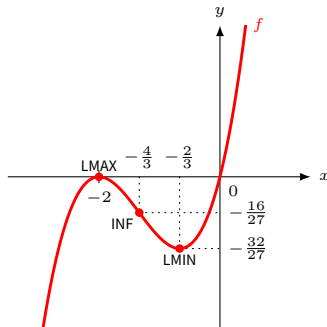
$$f''(x) = 6x + 8, \quad x \in \mathbb{R},$$

$$6x + 8 = 0 \quad \Rightarrow \quad x = -\frac{4}{3}$$





The function f is concave down on $(-\infty, -\frac{4}{3})$, it is concave up on $(-\frac{4}{3}, \infty)$. The function f has a point of inflection $x = -\frac{4}{3}$.





Example 3.63

Investigate the behaviour of the function $f(x) = 1 + \frac{2}{x} + \frac{1}{x^2}$.

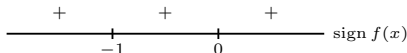
Solution We need to avoid division by zero, therefore the function f is defined for all $x \in (-\infty, 0) \cup (0, \infty)$. Since

$$f(-x) = 1 - \frac{2}{x} + \frac{1}{x^2}$$

the function f is not even, nor odd. We find the intercepts of the graph with axes and intervals on which the graph is below and above the x -axis.

$$1 + \frac{2}{x} + \frac{1}{x^2} = 0 \quad \Leftrightarrow \quad x^2 + 2x + 1 = 0 \quad \Leftrightarrow \quad (x + 1)^2 = 0 \quad \Leftrightarrow \quad x = -1$$

There is no intercept of the graph with the y -axis. The points $x = -1$, $x = 0$ divide the real line into three intervals and we have



The function f is nonnegative for all $x \neq 0$.



We find limits at the endpoints of the intervals $(-\infty, 0)$, $(0, \infty)$ given by the domain.

$$\lim_{x \rightarrow -\infty} \left(1 + \frac{2}{x} + \frac{1}{x^2} \right) = 1$$

$$\lim_{x \rightarrow +\infty} \left(1 + \frac{2}{x} + \frac{1}{x^2} \right) = 1$$

$$\lim_{x \rightarrow 0^-} \left(1 + \frac{2}{x} + \frac{1}{x^2} \right) = [1 - \infty + \infty] = \lim_{x \rightarrow 0^-} \frac{x^2 + 2x + 1}{x^2} = \left[\frac{1}{0^+} \right] = +\infty$$

$$\lim_{x \rightarrow 0^+} \left(1 + \frac{2}{x} + \frac{1}{x^2} \right) = [1 + \infty + \infty] = +\infty$$

The function f has a vertical asymptote $x = 0$. There is a horizontal asymptote at infinity and minus infinity $y = 1$.
We find the derivative

$$f'(x) = (1 + 2 \cdot x^{-1} + x^{-2})' = -2 \cdot x^{-2} - 2x^{-3} = -\frac{2}{x^2} - \frac{2}{x^3} = -\frac{2(x+1)}{x^3}$$



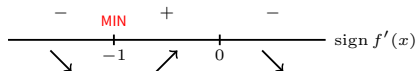
and its zero points

$$-\frac{2(x+1)}{x^3} = 0 \Rightarrow -2(x+1) = 0 \Rightarrow x = -1.$$

The points $x = -1, x = 0$ divide the real line into three intervals

$$(-\infty, -1), \quad (-1, 0), \quad (0, \infty).$$

Let's consider test points $c_1 = -2, c_2 = -\frac{1}{2}, c_3 = 1$ from these intervals. We get $f'(-2) < 0, f'(-\frac{1}{2}) > 0, f'(1) < 0$. We draw the following scheme.

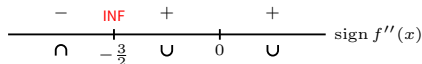


Function f is increasing on $[-1, 0)$ and decreasing on $(-\infty, -1], (0, \infty)$. There is no local extreme at $x = 0$, the value $f(-1) = 0$ is a local minimum. We find the second derivative and solve the equation $f''(x) = 0$.

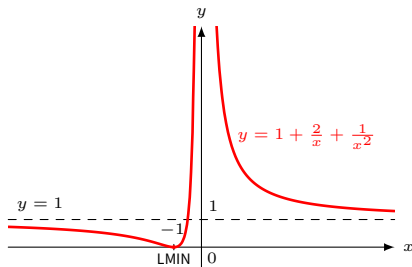
$$f''(x) = \left(-\frac{2}{x^2} - \frac{2}{x^3}\right)' = \frac{4}{x^3} + \frac{6}{x^4} = \frac{4x+6}{x^4}, \quad x \neq 0,$$



$$\frac{4x+6}{x^4} = 0 \Rightarrow 4x+6=0 \Rightarrow x = -\frac{3}{2}$$



The function f is concave down on $(-\infty, -\frac{3}{2})$, it is concave up on $(-\frac{3}{2}, 0)$ and $(0, \infty)$. The function f has a point of inflection $x = -\frac{3}{2}$.





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4. Indefinite Integral

We already know how to find the derivative of a function. Now, we will focus on the reverse of the derivative. For given function f defined on an interval I we want to find the function F on I which satisfies $F'(x) = f(x)$ for all x from the interior of I .



4.1 Indefinite Integral (Antiderivative)

Definition 4.1 (Hass, Giordano, Weir & Thomas, 2005)

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Example 4.2

The function $F(x) = \frac{1}{5}x^5$ is the antiderivative of the function $f(x) = x^4$ for all $x \in \mathbb{R}$.

Example 4.3

The function $F(x) = -x^{-1}$ is the antiderivative of the function $f(x) = x^{-2}$ for all $x \in (-\infty, 0)$ and $(0, \infty)$. This is not true for example in interval $(-2, 3)$, because this interval contains the point 0 which is not in the domain of the f .

Theorem 4.4 (Bouchala & Sadowská, 2007)

If a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous on an open interval $I \subset \mathbb{R}$, then f has an antiderivative in I .



The function $F(x) = \frac{1}{5}x^5$ is not the only function whose derivative is x^4 . The function $x^4 + 2$ has the same derivative. So does $x^4 + C$ for any constant C .

If F is an antiderivative of the function f on the interval I , then for any $C \in \mathbb{R}$ the function $G(x) = F(x) + C$ is also an antiderivative of f on I .

Definition 4.5 (Mařík, 2012)

Let f be a function defined on I . The set of all antiderivatives of the function f is said to be an **indefinite integral** of the function f on the interval I . The indefinite integral is denoted by

$$\int f(x)dx = F(x) + C.$$

The function which has an antiderivative (**indefinite integral**) on the interval I is said to be **integrable** on the interval I .

The symbol \int is called the **integral symbol**, $f(x)$ is called the **integrand**, x is called the integration variable and the C is called the **constant of integration**. The process of finding the indefinite integral is called **integration** or integrating $f(x)$.



If we need to be specific about the integration variable we will say that we are integrating $f(x)$ with respect to x . In speaking of an indefinite integral, we have always in mind some open interval too.

The dx indicates that the integral is to be taken with respect to x . In one of the previous section we called the dx a differential. It is necessary to use the dx at the end of the integral. The integral sign and the dx is something like a pair of parentheses. If you drop the dx it won't be clear where the integrand ends. You only integrate what is between the integral sign and the dx . See next examples.

$$\int x^2 + 4x + 1 \, dx = \frac{x^3}{3} + 2x^2 + x + C$$
$$\int x^2 + 4x \, dx + 1 = \frac{x^3}{3} + 2x^2 + C + 1$$

In the second integral the “+1” is outside the integral, it is left alone and not integrated. We get different answers. It is better to use brackets in the integral notation.

$$\int (x^2 + 4x + 1) \, dx = \frac{x^3}{3} + 2x^2 + x + C$$



4.2 Integration Formulas

From differentiation formulas we can derive formulas for antiderivatives. The following formulas hold on every open interval which forms a subset of the domain of the corresponding integrated function.

$$\int 0 \, dx = C$$

$$\int 1 \, dx = x + C$$

$$\int k \, dx = k \cdot x + C \quad (k \in \mathbb{R})$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{x} \, dx = \ln |x| + C$$

$$\int e^x \, dx = e^x + C$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$



$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \frac{1}{\cos^2 x} \, dx = \operatorname{tg} x + C$$

$$\int \frac{1}{\sin^2 x} \, dx = -\operatorname{cotg} x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = -\arccos x + C$$

$$\int \frac{1}{1+x^2} \, dx = \operatorname{arctg} x + C$$

$$\int \frac{1}{1+x^2} \, dx = -\operatorname{arccotg} x + C$$



$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C$$

$$\int \frac{1}{\sqrt{x^2 + a}} dx = \ln \left(x + \sqrt{x^2 + a} \right) + C \quad (a \neq 0)$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + C$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \cdot \operatorname{arctg} \frac{x}{a} + C$$

The previous table gives list of elementary integrals. The Theorem 4.4 says that the antiderivative exists for every continuous function on a closed interval but it does not show how to find it. This problem is more difficult than the calculation of the derivative. There are integrable elementary functions for which the antiderivative is known to exist, but it is not an elementary function.

For example the following integrals exist but cannot be written using elementary functions.

$$\int e^{-x^2} dx, \quad \int \frac{\sin x}{x} dx, \quad \int \frac{1}{\ln x} dx.$$



Example 4.6

Evaluate the following indefinite integrals

$$\text{a) } \int x^4 dx, \quad \text{b) } \int \sqrt{x} dx, \quad \text{c) } \int \frac{1}{\sqrt[4]{x^3}} dx.$$

Solution In each case, we use only the formula $\int x^n dx = \frac{x^{n+1}}{n+1} + C$.

$$\text{a) } \int x^4 dx = \frac{x^{4+1}}{4+1} = \frac{x^5}{5} + C,$$

$$\text{b) } \int \sqrt{x} dx = \int x^{\frac{1}{2}} dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}\sqrt{x^3} + C,$$

$$\text{c) } \int \frac{1}{\sqrt[4]{x^3}} dx = \int x^{-\frac{3}{4}} dx = \frac{x^{-\frac{3}{4}+1}}{-\frac{3}{4}+1} = \frac{x^{\frac{1}{4}}}{\frac{1}{4}} = 4\sqrt[4]{x} + C.$$



4.3 Properties of the Indefinite Integral

The constant multiple rule and the sum rule for differentiation lead to the following rules for the integration.

Theorem 4.7 (Linearity of the Indefinite Integral) (Mařík, 2012)

Let f , g be functions integrable on the interval I and c be a real number. The following relations hold on the interval I .

$$\int f(x) \pm g(x) \, dx = \int f(x) \, dx \pm \int g(x) \, dx,$$
$$\int c f(x) \, dx = c \int f(x) \, dx.$$

We can factor multiplicative constants out of the indefinite integrals. The integral of a sum or difference of functions is the sum or difference of the individual integrals.

There are no rules for integrals of the product $f(x) \cdot g(x)$ and the quotient $f(x)/g(x)$ or the composite function $f(g(x))$.



For $c_1, c_2, \dots, c_n \in \mathbb{R}$ and integrable functions f_1, f_2, \dots, f_n on the interval I the linearity rule for integration is in the form

$$\int (c_1 \cdot f(x) + c_2 \cdot f_2(x) + \dots + c_n \cdot f_n(x)) \, dx = c_1 \cdot \int f_1(x) \, dx + c_2 \cdot \int f_2(x) \, dx + \dots + c_n \cdot \int f_n(x) \, dx.$$

Example 4.8

Evaluate the integral $\int \left(4x^7 - 5x + \frac{2}{x} \right) \, dx$.

Solution We apply the linearity rule and integrate the individual terms using integration formulas.

$$\begin{aligned} \int \left(4x^7 - 5x + \frac{2}{x} \right) \, dx &= 4 \int x^7 \, dx - 5 \int x \, dx + 2 \int \frac{1}{x} \, dx = 4 \frac{x^8}{8} - 5 \frac{x^2}{2} + 2 \ln |x| = \\ &= \frac{x^8}{2} - \frac{5x^2}{2} + 2 \ln |x| + C. \end{aligned}$$



Example 4.9

Evaluate the integral $\int \frac{(x + \sqrt{x})^2}{\sqrt[3]{x}} dx$.

Solution There is no “Quotient rule” for integrals. We need to rewrite the integrand to obtain elementary integrals. In the numerator we can multiply out the power and break the fraction into smaller fractions

$$\int \frac{(x + \sqrt{x})^2}{\sqrt[3]{x}} dx = \int \frac{x^2 + 2x\sqrt{x} + x}{\sqrt[3]{x}} dx = \int \frac{x^2}{\sqrt[3]{x}} dx + 2 \int \frac{x\sqrt{x}}{\sqrt[3]{x}} dx + \int \frac{x}{\sqrt[3]{x}} dx.$$

We convert the roots to fractional exponents and get rid of the fractions by dividing.

$$\begin{aligned} \int \frac{x^2}{\sqrt[3]{x}} dx + 2 \int \frac{x\sqrt{x}}{\sqrt[3]{x}} dx + \int \frac{x}{\sqrt[3]{x}} dx &= \int x^2 x^{-\frac{1}{3}} dx + 2 \int x x^{\frac{1}{2}} x^{-\frac{1}{3}} dx + \int x x^{-\frac{1}{3}} dx \\ &= \int x^{\frac{5}{3}} dx + 2 \int x^{\frac{7}{6}} dx + \int x^{\frac{2}{3}} dx. \end{aligned}$$



Now, we have only elementary integrals to evaluate.

$$\int x^{\frac{5}{3}} dx + 2 \int x^{\frac{7}{6}} dx + \int x^{\frac{2}{3}} dx = \frac{x^{\frac{8}{3}}}{\frac{8}{3}} + 2 \frac{x^{\frac{13}{6}}}{\frac{13}{6}} + \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + C.$$

The last thing is to rewrite the terms. We convert a fractional exponent to the root.

$$\int \frac{(x + \sqrt{x})^2}{\sqrt[3]{x}} dx = \frac{3\sqrt[3]{x^8}}{8} + \frac{12\sqrt[6]{x^{13}}}{13} + \frac{3\sqrt[3]{x^5}}{5} + C.$$

We need to determine validity of this result. Due the term with the square root and the quotient in the given integral it must be $x > 0$. The same condition is valid also for the result. So, our calculation is valid only for $x > 0$.

Example 4.10

Evaluate the integral $\int \tan^2 x dx$.

Solution We integrate a square of the tangent function and there is no integration rule for this function.



We can use trig identities

$$\tan x = \frac{\sin x}{\cos x}, \quad \sin^2 x + \cos^2 x = 1,$$

that change integrand into the suitable form.

$$\begin{aligned} \int \tan^2 x \, dx &= \int \frac{\sin^2 x}{\cos^2 x} \, dx = \int \frac{1 - \cos^2 x}{\cos^2 x} \, dx = \int \left(\frac{1}{\cos^2 x} - 1 \right) \, dx = \\ &= \int \frac{1}{\cos^2 x} \, dx - \int 1 \, dx = \tan x - x + C. \end{aligned}$$

We further simplified the integrand by breaking the fraction up.



4.4 Integration by Parts

We can not integrate product of functions in the same way that we integrate sums and differences. Besides, every product is different, so there is no rule that will work for all products.

Theorem 4.11 (Integration by Parts) (Bouchala & Sadowská, 2007)

Let functions u and v have continuous first derivatives on an open interval I . Then

$$\int u(x)v'(x) \, dx = u(x)v(x) - \int u'(x)v(x) \, dx \quad \text{on } I. \quad (3)$$

Integration by parts is one of the technique for integrals of the products. It is derived from the product rule for differentiation.

$$\begin{aligned} [u(x)v(x)]' &= u'(x)v(x) + u(x)v'(x) \quad \Rightarrow \\ u(x)v(x) &= \int u'(x)v(x) \, dx + \int u(x)v'(x) \, dx. \end{aligned}$$

Rearranging the terms of this last equation, we get the **integration by parts** formula.



Let $P(x)$ be a polynomial. We use the integration by parts in the evaluation of the integrals like

$$\int P(x)e^{ax} dx, \int P(x) \sin(ax) dx, \int P(x) \cos(ax) dx$$

and

$$\int P(x) \operatorname{arctg} x dx, \int P(x) \ln^k x dx,$$

where $a \in \mathbb{R}$, $k \in \mathbb{N}$. In the first set of the integrals the polynomial function is differentiated to decrease its degree. We can repeat this process as needed. In the second set of the integrals we differentiate the functions $\operatorname{arctg} x$ and $\ln x$.

Example 4.12

Evaluate the integral $\int x \cdot \sin x dx$.

Solution We need to express the integrated function as a product of two functions denoted $u(x)$ and $v'(x)$. In our case we have $u(x) = x$, $v'(x) = \sin x$. We determine $u'(x)$ by differentiating the expression $u(x)$ and $v(x)$ by integrating $v'(x)$, no constant of integration is used.



$$u'(x) = (x)' = 1, \quad v(x) = \int \sin x \, dx = -\cos x.$$

We plug all expressions into the formula (3).

$$\int x \cdot \sin x \, dx = x \cdot (-\cos x) - \int (-\cos x) \cdot 1 \, dx = -x \cos x + \sin x + C.$$

Example 4.13

Evaluate the integral $\int x^2 \cdot e^x \, dx$.

Solution We use the formula (3) with $u = x^2$, $v' = e^x$, then $u' = 2x$, $v = e^x$. We usually write the integration by parts into a matrix.

$$\int x^2 \cdot e^x \, dx = \begin{vmatrix} u = x^2 & v' = e^x \\ u' = 2x & v = e^x \end{vmatrix} = x^2 e^x - \int 2x e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$



The new integral is similar to the initial integral. We can do integration by parts again with $u = x$ and $v' = e^x$.

$$\left| \begin{array}{ll} u = x & v' = e^x \\ u' = 1 & v = e^x \end{array} \right| = x^2 e^x - 2 \left(x e^x - \int e^x dx \right) = x^2 e^x - 2x e^x + 2e^x + C.$$

Example 4.14

Evaluate the integral $\int x^3 \cdot \ln 2x dx$.

Solution Both of two previous examples took u to be the polynomial. Now, we choose $u = \ln 2x$ and $v' = x^3$.

$$\begin{aligned} \int x^3 \cdot \ln 2x dx &= \left| \begin{array}{ll} u = \ln 2x & v' = x^3 \\ u' = \frac{1}{x} & v = \frac{x^4}{4} \end{array} \right| = \frac{x^4}{4} \ln 2x - \int \frac{1}{x} \cdot \frac{x^4}{4} dx = \\ &= \frac{x^4}{4} \ln 2x - \frac{1}{4} \int x^3 dx = \frac{x^4}{4} \ln 2x - \frac{1}{4} \cdot \frac{x^4}{4} = \frac{x^4}{4} \ln 2x - \frac{x^4}{16} + C. \end{aligned}$$



Example 4.15

Evaluate the integral $\int e^x \sin 3x \, dx$.

Solution In this case it does not matter which we choose to be u . We can put the exponential in either the u or the v' and the sine in the other. Since we do not know at this moment how to integrate $\sin 3x$, let us put $u = \sin 3x$ and $v' = e^x$.

$$\int e^x \sin 3x \, dx = \left| \begin{array}{ll} u = \sin 3x & v' = e^x \\ u' = 3 \cos 3x & v = e^x \end{array} \right| = e^x \sin 3x - 3 \int e^x \cos 3x \, dx.$$

We integrate by parts again.

$$\begin{aligned} \left| \begin{array}{ll} u = \cos 3x & v' = e^x \\ u' = -3 \sin 3x & v = e^x \end{array} \right| &= e^x \sin 3x - 3 \left(e^x \cos 3x + 3 \int e^x \sin 3x \, dx \right) = \\ &= e^x \sin 3x - 3e^x \cos 3x - 9 \int e^x \sin 3x \, dx. \end{aligned}$$



The integral is now

$$\int e^x \sin 3x \, dx = e^x \sin 3x - 3e^x \cos 3x - 9 \int e^x \sin 3x \, dx.$$

We have the same integral on both sides of the equal sign. On the right side it is multiplied by minus nine. We can add the integral multiplied by nine to both sides to get

$$10 \int e^x \sin 3x \, dx = e^x \sin 3x - 3e^x \cos 3x.$$

We divide the equation by 10 and add the constant of integration. The integral is

$$\int e^x \sin 3x \, dx = \frac{1}{10} (e^x \sin 3x - 3e^x \cos 3x) + C.$$



4.5 First Substitution Rule

Let us start with the simple example. We can not integrate

$$\int 2 \cos 2x \, dx.$$

If we let $t = 2x$ and we compute the differential for this, we get

$$dt = 2dx.$$

We can eliminate every x that exists in the integral and write the integral completely in terms of t using both the definition of t and its differential.

$$\int 2 \cos 2x \, dx = \int \cos 2x \, 2dx = \int \cos t \, dt.$$

We evaluate this integral and get the integral back into terms of the original variable.

$$\int 2 \cos 2x \, dx = \int \cos t \, dt = \sin t = \sin 2x + C.$$

The process above is called the **Substitution Rule**.



Theorem 4.16 (First Substitution Rule) (Hass, Giordano, Weir & Thomas, 2005)

Let $t = g(x)$ be a differentiable function on J whose range is an interval I , and f be continuous on I , then

$$\int f(g(x)) g'(x) dx = \left| \begin{array}{l} t = g(x) \\ dt = g'(x) dx \end{array} \right| = \int f(t) dt = F(t) = F(g(x)) \quad \text{on } J.$$

The statement follows directly from the Chain Rule

$$(F(g(x)))' = F'(g(x))g'(x) = f(g(x))g'(x) \quad (\text{on } J).$$

It can be difficult to identify the correct substitution. We need to have an integrand in the form

$$f(g(x)) \cdot g'(x).$$

It is a product of a composite function $f(g(x))$ with the derivative of its inside function $g(x)$ except for a multiplicative constant. Upon the substitution $t = g(x)$ every x in the integral (including also dx) must disappear from the integral.



Example 4.17

Evaluate the integral $\int \sin^2 x \cdot \cos x \, dx$.

Solution Let us put $t = \sin x$ and compute the differential

$$dt = \cos x \, dx.$$

We write t instead of $\sin x$ and dt instead of $\cos x \, dx$ in the integral.

$$\int \sin^2 x \cdot \cos x \, dx = \left| \begin{array}{l} t = \sin x \\ dt = \cos x \, dx \end{array} \right| = \int t^2 \, dt = \frac{t^3}{3} = \frac{\sin^3 x}{3} + C.$$

Don't forget to return back to the original variable.

Example 4.18

Evaluate the integral $\int \sqrt{4x+2} \, dx$.

Solution A substitution $t = 4x + 2$ leads to the equality $dt = 4 \, dx$. The constant factor 4 is missing from the integral.



We can rearrange this equality to $\frac{1}{4} dt = dx$ and then replace dx in the integral.

$$\int \sqrt{4x+2} dx = \left| \begin{array}{lcl} t & = & 4x+2 \\ dt & = & 4 dx \\ \frac{1}{4} dt & = & dx \end{array} \right| = \int t^{\frac{1}{2}} \frac{1}{4} dt = \frac{1}{4} \cdot \frac{t^{\frac{3}{2}}}{\frac{3}{2}} = \frac{\sqrt{(4x+2)^3}}{6} + C.$$

Example 4.19

Evaluate the integral $\int x^4 (2+x^5)^{10} dx$.

Solution We can expand $x^4 (2+x^5)^{10}$ through the binomial formula in the form of a polynomial function which is easy to integrate. To save time it is better to use a substitution. Let us consider $t = 2+x^5$, it leads to $dt = 5x^4 dx$. We have got an x^4 out in front of the parenthesis but we do not have a 5. Besides the method from the previous example, we can introduce this factor after the integral sign if we compensate for it by a factor of $1/5$ in front of the integral sign.

$$\int x^4 (2+x^5)^{10} dx = \frac{1}{5} \int (2+x^5)^{10} 5x^4 dx = \left| \begin{array}{lcl} t & = & 2+x^5 \\ dt & = & 5x^4 dx \end{array} \right| =$$



$$= \frac{1}{5} \int t^{10} dt = \frac{1}{5} \cdot \frac{t^{11}}{11} = \frac{(2 + x^5)^{11}}{55} + C.$$

Example 4.20

Evaluate the integral $\int \frac{1}{x(3 \ln x + 5)^4} dx$.

Solution We can use $t = \ln x$ or even $t = 3 \ln x + 5$.

$$\begin{aligned} \int \frac{1}{x(3 \ln x + 5)^4} dx &= \left| \begin{array}{l} t = 3 \ln x + 5 \\ dt = \frac{3}{x} dx \\ 3 dt = \frac{1}{x} dx \end{array} \right| = \int \frac{1}{t^4} 3 dt = 3 \int t^{-4} dt = \\ &= 3 \cdot \frac{t^{-3}}{-3} = -\frac{1}{(3 \ln x + 5)^3} + C. \end{aligned}$$



A table of basic integration formulas (the section 4.2) contains the formula

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C.$$

It is derived using a substitution method with $t = f(x)$, $dt = f'(x)dx$.

$$\int \frac{f'(x)}{f(x)} dx = \left| \begin{array}{l} t = f(x) \\ dt = f'(x)dx \end{array} \right| = \int \frac{1}{t} dt = \ln |t| = \ln |f(x)| + C.$$

Example 4.21

Evaluate the integral $\int \frac{e^x}{e^x - 5} dx$.

Solution In order to use above mentioned formula the numerator need to be a derivative of the denominator except for a multiplicative constant. It is our case, so the integral is

$$\int \frac{e^x}{e^x - 5} dx = \ln |e^x - 5| + C.$$



Example 4.22

Evaluate the integral $\int \frac{x-3}{x^2-6x+5} dx$.

Solution The numerator differs from the derivative of the denominator only by a multiplicative constant.

$$\int \frac{x-3}{x^2-6x+5} dx = \frac{1}{2} \int \frac{2x-6}{x^2-6x+5} dx = \frac{1}{2} \ln|x^2-6x+5| + C.$$

Some integrals can be evaluated using several different techniques. Different methods can lead to different results but their difference is only a constant. The following problems need to be solved by the substitution method in connection with the integration by parts method.

Example 4.23

Evaluate the integral $\int e^{\sqrt{x}} dx$.

Solution Let's make the substitution $t = \sqrt{x} = x^{\frac{1}{2}}$. For this t we have



$$\begin{aligned} dt &= \frac{1}{2} x^{-\frac{1}{2}} dx = \frac{1}{2\sqrt{x}} dx \\ 2\sqrt{x} dt &= dx \\ 2t dt &= dx. \end{aligned}$$

The substitution leads to the integral

$$\int e^{\sqrt{x}} dx = 2 \int t \cdot e^t dt.$$

Now, we do integration by parts.

$$\int t \cdot e^t dt = \left| \begin{array}{ll} u = t & v' = e^t \\ u' = 1 & v = e^t \end{array} \right| = t \cdot e^t - \int e^t dt = t \cdot e^t - e^t.$$

Finally,

$$\int e^{\sqrt{x}} dx = 2 \cdot \left(\sqrt{x} \cdot e^{\sqrt{x}} - e^{\sqrt{x}} \right) + C.$$



Example 4.24

Evaluate the integral $\int \arcsin x \, dx$.

Solution The function arcsine is missing in the table of elementary integrals because this function is not a derivative of any elementary function. An integral of arcsine can be evaluated using integration by parts, although there is only a single function in the integral. Let's use

$$u = \arcsin x, \quad v' = 1.$$

Then, the integral is

$$\int 1 \cdot \arcsin x \, dx = \left| \begin{array}{ll} u = \arcsin x & v' = 1 \\ u' = \frac{1}{\sqrt{1-x^2}} & v = x \end{array} \right| = x \arcsin x - \int x \frac{1}{\sqrt{1-x^2}} \, dx.$$

Now, we will bring the root up to the numerator and change it into fractional exponent form. The substitution $t = 1 - x^2$, $dt = -2x \, dx$ will ensue.

$$\int \arcsin x \, dx = x \arcsin x - \int x \frac{1}{\sqrt{1-x^2}} \, dx = x \arcsin x - \int x (1-x^2)^{-\frac{1}{2}} \, dx =$$



$$\begin{aligned} &= \left| \begin{array}{rcl} t & = & 1 - x^2 \\ dt & = & -2x \, dx \\ -\frac{1}{2} dt & = & x \, dx \end{array} \right| = x \arcsin x + \frac{1}{2} \int t^{-\frac{1}{2}} dt = \\ &= x \arcsin x + \frac{1}{2} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = \\ &= x \arcsin x + \sqrt{1 - x^2} + C. \end{aligned}$$

The integration holds for $x \in [-1, 1]$.



4.6 Second Substitution Rule

Theorem 4.25 (Second Substitution Rule) (Bouchala & Sadowská, 2007)

Let a function f be continuous on I , let a function $\varphi : J \rightarrow I$ have a continuous and non-zero derivative in J . Then

$$\int f(x) dx = \left| \begin{array}{l} x = \varphi(t) \\ dx = \varphi'(t) dt \end{array} \right| = \int f(\varphi(t)) \varphi'(t) dt \quad \text{on } J.$$

We simplify an integral by replacing x with $\varphi(t)$ and by replacing dx by $\varphi'(t) dt$. We can also substitute $t = \varphi^{-1}(x)$, where φ^{-1} denotes the inverse function to the function φ . Its existence follows from the fact that the derivative of the function φ is non-zero.

Example 4.26

Evaluate the integral $\int \sqrt{9 - x^2} dx$, $x \in (-3, 3)$.

Solution The substitution is $x = 3 \sin t$, $dx = 3 \cos t dt$.



$$\begin{aligned}\int \sqrt{9-x^2} dx &\stackrel{\text{subst.}}{=} \int \sqrt{9-9\sin^2 t} \cdot 3 \cos t dt = \\&= \int \sqrt{9 \cdot (1-\sin^2 t)} \cdot 3 \cos t dt = 9 \cdot \int \sqrt{\cos^2 t} \cdot \cos t dt = 9 \cdot \int |\cos t| \cdot \cos t dt = \\&= 9 \cdot \int \cos^2 t dt = 9 \cdot \int \frac{1+\cos 2t}{2} dt = \frac{9}{2} \cdot \int (1+\cos 2t) dt = \\&= \frac{9}{2} \left(t + \frac{\sin 2t}{2} \right) = \frac{9}{2} (t + \sin t \cos t) = \frac{9}{2} \left(t + \sin t \cdot \sqrt{1-\sin^2 t} \right) + C = \\&= \frac{9}{2} \cdot \left(\arcsin \frac{x}{3} + \frac{x}{3} \cdot \sqrt{1-\left(\frac{x}{3}\right)^2} \right) + C = \frac{9}{2} \arcsin \frac{x}{3} + \frac{x}{2} \cdot \sqrt{9-x^2} + C,\end{aligned}$$

where

$$\int \cos 2t dt = \left| \begin{array}{lcl} 2t & = & u \\ 2 dt & = & du \end{array} \right| = \frac{1}{2} \cdot \int \cos u du = \frac{\sin 2t}{2}.$$

If $x \in (-3, 3)$, then $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\cos t > 0$, so $|\cos t| = +\cos t$.



4.7 Integration of Rational Functions

A rational function is by definition the quotient of two polynomials

$$R(x) = \frac{P_n(x)}{Q_m(x)},$$

where $P_n(x)$ is an n -degree polynomial and $Q_m(x)$ is an m -degree polynomial. For example

$$\frac{1}{x^2 - 4}, \quad \frac{x^4}{x^2 + 1}, \quad \frac{x + \sqrt{3}}{x^3 + 3x^2 + 3x + 1}$$

are all rational functions.

Definition 4.27 (Mařík, 2012)

Let $R(x) = \frac{P_n(x)}{Q_m(x)}$ be a rational function. The function $R(x)$ is said to be **proper** if $n < m$ and **improper** otherwise.

Theorem 4.28 (“Math Tutor”, 2019)

Every rational function can be written as a sum of a polynomial and a proper rational function with the same denominator.



Example 4.29

Proper and improper rational functions:

- ▶ $\frac{3x^2 + 2x + 4}{x^4 + 1}$ – proper rational function
- ▶ $\frac{x^4 - 2x}{x^2 + x + 1}$ – improper rational function
- ▶ $\frac{x^2 + x}{4x^2 + 1}$ – improper rational function

If the degree of the denominator is less than the degree of the numerator, we perform polynomial long division with remainder to decompose improper rational function according to Theorem 4.28.

Example 4.30

Use polynomial long division to rewrite $R(x) = \frac{3x^4 - 2x^2 + 5}{x^2 + 1}$.



Solution First we divide the leading term $3x^4$ of the numerator polynomial by the leading term x^2 of the divisor, and write the answer $3x^2$ to right. Then, we multiply this term $3x^2$ by the divisor $x^2 + 1$ and write the answer under the numerator polynomial.

$$\begin{array}{r} (3x^4 - 2x^2 + 5) : (x^2 + 1) = 3x^2 - 5 \\ - (3x^4 + 3x^2) \\ \hline -5x^2 + 5 \\ - (-5x^2 - 5) \\ \hline 10 \end{array}$$

The calculation is done, if the remainder is of lower degree than the denominator. Otherwise, we divide the leading term of the remainder by the leading term of the denominator and so on. Finally

$$\frac{3x^4 - 2x^2 + 5}{x^2 + 1} = 3x^2 - 5 + \frac{10}{x^2 + 1}.$$



A polynomial can be integrated by basic formulas and the proper rational function need to be decomposed into a sum of simpler fractions, called **partial fractions**, which are easily integrated. For instance, the rational function

$\frac{6x - 10}{x^2 - 4x + 3}$ can be rewritten as

$$\frac{6x - 10}{x^2 - 4x + 3} = \frac{2}{x - 1} + \frac{4}{x - 3}.$$

Then the integral is simple.

$$\int \frac{6x - 10}{x^2 - 4x + 3} dx = \int \frac{2}{x - 1} dx + \int \frac{4}{x - 3} dx = 2 \ln |x - 1| + 4 \ln |x - 3| + C.$$

The method of rewriting rational functions as a sum of simpler fractions is called **Partial Fractions Decomposition**.

At first, we need to factor the denominator into irreducible polynomials, it means into a product of powers of linear factors and powers of irreducible (not further decomposable to linear factors) quadratic factors. Then for each factor in the denominator we determine the term or terms we pick up in the partial fraction decomposition.



Let $R(x) = \frac{P_n(x)}{Q_m(x)}$ be a proper rational function. Suppose that the polynomials $P_n(x)$ and $Q_m(x)$ have no common zeros.

- Let us assign to each simple real root x_0 of the polynomial $Q_m(x)$ the fraction

$$\frac{A}{x - x_0},$$

where A is some (not yet determined) real constant.

- Let us assign to each real root x_0 of the multiplicity k of the polynomial $Q_m(x)$ the k -tuple of the fractions

$$\frac{A_1}{x - x_0}, \quad \frac{A_2}{(x - x_0)^2}, \quad \dots, \quad \frac{A_k}{(x - x_0)^k},$$

where A_i are some real constants.

- An irreducible quadratic factor $(x^2 + Mx + N)$ of the polynomial $Q_m(x)$ has a pair of the mutually conjugate complex roots. Let us assign to this pair of complex roots the fraction

$$\frac{Bx + C}{x^2 + Mx + N},$$

where B and C are some (not yet determined) real numbers.



- Let us assign to irreducible quadratic factor $(x^2 + Mx + N)^k$ of the denominator $Q_m(x)$ the k -tuple of the fractions

$$\frac{B_1x + C_1}{x^2 + Mx + N}, \quad \frac{B_2x + C_2}{(x^2 + Mx + N)^2}, \quad \dots, \quad \frac{B_kx + C_k}{(x^2 + Mx + N)^k}.$$

All the constants A, B, C, A_i, B_i are uniquely determined. It means that the function $R(x)$ can be written as a sum of all of the fractions considered above in just one way up to the order in the sum.

Example 4.31

- $\frac{7x - 1}{2x^2 + 3x - 2} = \frac{7x - 1}{(x + 2)(2x - 1)} = \frac{A}{x + 2} + \frac{B}{2x - 1}$
- $\frac{3x^2 + 2x - 11}{(x + 4)(x - 3)^3} = \frac{A}{x + 4} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{D}{(x - 3)^3}$
- $\frac{2x^3 - 2x + 1}{x^2(x^2 - x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 - x + 1}$



We should already be able to integrate the partial fractions using basic integration formulas and rules. Partial fraction with linear term can be integrated using linear substitution.

$$\int \frac{1}{x - x_0} dx = \left| \begin{array}{l} t = x - x_0 \\ dt = dx \end{array} \right| = \int \frac{1}{t} dt = \ln |t| = \ln |x - x_0| + C,$$

$$\begin{aligned} \int \frac{1}{(x - x_0)^k} dx &= \left| \begin{array}{l} t = x - x_0 \\ dt = dx \end{array} \right| = \int \frac{1}{t^k} dt = \int t^{-k} dt = \frac{t^{-k+1}}{-k+1} = \\ &= \frac{1}{1-k} \cdot \frac{1}{(x - x_0)^{k-1}} + C. \end{aligned}$$

An integration of partial fractions with quadratic terms $x^2 + a^2$ as denominator and a constant in the numerator leads to arctangent.

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \cdot \operatorname{arctg} \frac{x}{a} + C,$$

$$\int \frac{1}{(x + b)^2 + a^2} dx = \left| \begin{array}{l} t = x + b \\ dt = dx \end{array} \right| = \int \frac{1}{t^2 + a^2} dt = \frac{1}{a} \cdot \operatorname{arctg} \frac{x + b}{a} + C.$$



Partial fraction with x in the numerator can be integrated using a quadratic substitution or we can rearrange the nominator to be a derivative of the denominator.

$$\int \frac{x}{x^2 + a^2} dx = \frac{1}{2} \int \frac{2x}{x^2 + a^2} dx = \frac{1}{2} \ln |x^2 + a^2|.$$

The following partial fraction contains quadratic polynomial of the type $x^2 + a^2$ in the denominator and a linear term in the numerator. This type of partial fraction can be split into two fractions that we already know how to integrate.

$$\int \frac{Ax + B}{x^2 + a^2} dx = \frac{A}{2} \int \frac{2x}{x^2 + a^2} dx + B \int \frac{1}{x^2 + a^2} dx.$$

Partial fractions with a general quadratic term in the denominator

$$\int \frac{Ax + B}{x^2 + Mx + N}$$

can be integrated using completing square in the denominator and suitable substitution.

For simplicity, we will consider the rational functions without a higher power of a quadratic term in the denominator only (the denominator has not multiple complex roots).



Let us summarize the method of the integration of the rational functions $R(x) = \frac{P_n(x)}{Q_m(x)}$.

1. If the degree of the denominator is less than the degree of the numerator, we perform polynomial long division with remainder to decompose improper rational function into a sum of a polynomial and a proper rational function.
2. We factor the denominator $Q_m(x)$ into irreducible polynomials.
3. We decompose the proper rational function into a sum of partial fractions. For the factor $x - x_0$ we add the fraction of the type $\frac{A}{x - x_0}$ to the decomposition. For the factor $(x - x_0)^k$ we add the fractions $\frac{A_1}{x - x_0}$, $\frac{A_2}{(x - x_0)^2}, \dots, \frac{A_k}{(x - x_0)^k}$. For every factor $x^2 + Mx + N$ we add the fraction $\frac{Bx + C}{x^2 + Mx + N}$ to the decomposition. The number of unknown constants is the same as the degree of the denominator. There are several methods for determining these constants, we will demonstrate them in the examples.
4. We integrate a polynomial from the step one and also partial fractions.



Example 4.32

Evaluate the integral $\int \frac{3}{(2x+5)^4} dx$.

Solution We integrate the partial fraction with linear term in the denominator.

$$\begin{aligned} \int \frac{3}{(2x+5)^4} dx &= \left| \begin{array}{l} t = 2x+5 \\ dt = 2 dx \end{array} \right| = \frac{3}{2} \int \frac{2}{(2x+5)^4} dx = \frac{3}{2} \int \frac{1}{t^4} dt = \\ &= \frac{3}{2} \int t^{-4} dt = \frac{3}{2} \cdot \frac{t^{-3}}{-3} = \frac{-1}{2(2x+5)^3} + C. \end{aligned}$$

Example 4.33

Evaluate the integral $\int \frac{x}{x^2+2x+5} dx$.

Solution The quadratic term in the denominator has no real roots. We need to create the derivative of x^2+2x+5 in the numerator. Since $(x^2+2x+5)' = 2x+2$, we have



$$x = \frac{1}{2} \cdot 2x = \frac{1}{2} \cdot (2x + 2 - 2) = \frac{1}{2} \cdot (2x + 2) - 1.$$

We rewrite the integrand

$$\frac{x}{x^2 + 2x + 5} = \frac{1}{2} \frac{2x + 2}{x^2 + 2x + 5} - \frac{1}{x^2 + 2x + 5}$$

and integrate the sum term by term and factor out constants:

$$\int \frac{x}{x^2 + 2x + 5} dx = \frac{1}{2} \int \frac{2x + 2}{x^2 + 2x + 5} dx - \int \frac{1}{x^2 + 2x + 5} dx.$$

For the integrand $\frac{2x + 2}{x^2 + 2x + 5}$ we substitute $t = x^2 + 2x + 5$ and $dt = (2x + 2)dx$. For the integrand $\frac{1}{x^2 + 2x + 5}$ we complete the square in the denominator and then use substitution.

$$\int \frac{x}{x^2 + 2x + 5} dx = \frac{1}{2} \int \frac{1}{t} dt - \int \frac{1}{(x + 1)^2 + 4} dx = \left| \begin{array}{lcl} s & = & x + 1 \\ ds & = & dx \end{array} \right| =$$



$$\begin{aligned} &= \frac{1}{2} \ln |t| - \int \frac{1}{s^2 + 4} ds = \frac{1}{2} \ln |t| - \frac{1}{2} \cdot \operatorname{arctg} \frac{s}{2} + C = \\ &= \frac{1}{2} \ln |x^2 + 2x + 5| - \frac{1}{2} \cdot \operatorname{arctg} \frac{x+1}{2} + C. \end{aligned}$$

Example 4.34

Evaluate the integral $\int \frac{8x-1}{x^2+x-6} dx$.

Solution The first step is to factor the denominator as $(x-2)(x+3)$ and the decomposition into partial fractions with undetermined coefficients.

$$\frac{8x-1}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}.$$

We clear the fractions by multiplying by the common denominator $(x-2)(x+3)$ (the denominator of the left hand side).

$$8x-1 = A(x+3) + B(x-2).$$



This equality is supposed to be true for all x . In particular it must be true for $x = 2$ and $x = -3$ which are the roots of linear factors. If we substitute $x = 2$, we get

$$8 \cdot 2 - 1 = A(2 + 3) + B(2 - 2) \Rightarrow 15 = 5A + 0 \Rightarrow A = 3,$$

and by substitution $x = -3$, we get

$$8 \cdot (-3) - 1 = A(-3 + 3) + B(-3 - 2) \Rightarrow -25 = 0 - 5B \Rightarrow B = 5.$$

We have all coefficients, so the integral is

$$\int \frac{8x - 1}{x^2 + x - 6} dx = \int \frac{3}{x - 2} + \frac{5}{x + 3} dx = 3 \ln |x - 2| + 5 \ln |x + 3| + C.$$

To do this we first split it up into two integrals and then used the substitutions $t = x - 2$, $s = x + 3$ on the integrals.

Example 4.35

Evaluate the integral $\int \frac{x^2 + 2}{x^3 - 2x^2 + x} dx$.

Solution We factor the denominator at first.



$$\frac{x^2 + 2}{x^3 - 2x^2 + x} = \frac{x^2 + 2}{x \cdot (x^2 - 2x + 1)} = \frac{x^2 + 2}{x \cdot (x - 1)^2}.$$

We express the integrand as a sum of partial fractions.

$$\frac{x^2 + 2}{x \cdot (x - 1)^2} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2}.$$

We multiply this equality by a common denominator $x \cdot (x - 1)^2$.

$$x^2 + 2 = A(x - 1)^2 + Bx(x - 1) + Cx. \quad (*)$$

The substitutions $x = 0, x = 1$ lead to

$$0^2 + 2 = A(0 - 1)^2 + B \cdot 0 \cdot (0 - 1) + C \cdot 0 \quad \Rightarrow \quad A = 2,$$

$$1^2 + 2 = A(1 - 1)^2 + B \cdot 1 \cdot (1 - 1) + C \cdot 1 \quad \Rightarrow \quad C = 3.$$

We have to find the value of the constant B . We can substitute an arbitrary number for x in the equality $(*)$.



If we substitute into (*) something different from $x = 0, x = 1$, say $x = -1$, we obtain an equation for B .

$$(-1)^2 + 2 = A(-1 - 1)^2 + B \cdot (-1) \cdot (-1 - 1) + C \cdot (-1)$$

$$3 = 4A + 2B - C$$

$$3 = 4 \cdot 2 + 2B - 3$$

$$-2 = 2B \Rightarrow B = -1.$$

We can integrate now.

$$\begin{aligned} \int \frac{x^2 + 2}{x^3 - 2x^2 + x} dx &= \int \frac{2}{x} dx + \int \frac{-1}{x-1} dx + \int \frac{3}{(x-1)^2} dx = \left| \begin{array}{l} t = x-1 \\ dt = dx \end{array} \right| = \\ &= 2 \int \frac{1}{x} dx - \int \frac{1}{t} dt + 3 \int \frac{1}{t^2} dt = 2 \ln |x| - \ln |t| + 3 \frac{t^{-1}}{-1} + C = \\ &= 2 \ln |x| - \ln |x-1| - \frac{3}{x-1} + C. \end{aligned}$$



Example 4.36

Evaluate the integral $\int \frac{3x^2 - x + 18}{x(x^2 + 9)} dx$.

Solution The denominator is already factored, so we can do the partial fraction decomposition.

$$\frac{3x^2 - x + 18}{x(x^2 + 9)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 9}.$$

We multiply both sides by $x(x^2 + 9)$ and get

$$3x^2 - x + 18 = A(x^2 + 9) + (Bx + C)x. \quad (**)$$

After substitution $x = 0$ we obtain

$$3 \cdot 0^2 - 0 + 18 = A(0^2 + 9) + (B \cdot 0 + C) \cdot 0 \Rightarrow 18 = 9A \Rightarrow A = 2.$$

We can substitute an arbitrary number for x in the equality $(**)$ or use the different way. Let us multiply out the right side of $(**)$

$$3x^2 - x + 18 = Ax^2 + 9A + Bx^2 + Cx$$



and collect all the like powers of x together.

$$3x^2 - x + 18 = (A + B)x^2 + Cx + 9A.$$

Equating the coefficients of like powers of x we obtain the following system of equations

$$\begin{aligned}x^2 : \quad 3 &= A + B, \\x^1 : \quad -1 &= C, \\x^0 : \quad 18 &= 9A.\end{aligned}$$

From the third equation we have $A = 2$ again, from the second one we have $C = -1$. Using known value of A in the first equation we get $B = 1$. We can integrate now.

$$\begin{aligned}\int \frac{3x^2 - x + 18}{x(x^2 + 9)} dx &= \int \frac{2}{x} dx + \int \frac{x - 1}{x^2 + 9} dx = 2 \int \frac{1}{x} dx + \int \frac{x}{x^2 + 9} dx \\&\quad - \int \frac{1}{x^2 + 9} dx = 2 \int \frac{1}{x} dx + \frac{1}{2} \int \frac{2x}{x^2 + 9} dx - \int \frac{1}{x^2 + 9} dx = \\&= 2 \ln |x| + \frac{1}{2} \ln (x^2 + 9) - \frac{1}{3} \arctan \frac{x}{3} + C.\end{aligned}$$



Example 4.37

Evaluate the integral $\int \frac{x^4 + 2x^3 + 2x^2 + 10x - 3}{x^3 + 3x} dx$.

Solution The rational function is not proper. The degree of the denominator is less than the degree of the numerator. We need to do long division to get a polynomial plus a proper fraction.

$$\begin{array}{r} (x^4 + 2x^3 + 2x^2 + 10x - 3) : (x^3 + 3x) = x + 2 \\ -(x^4 + 3x^2) \\ \hline 2x^3 - x^2 + 10x - 3 \\ -(2x^3 + 6x) \\ \hline -x^2 + 4x - 3 \end{array}$$

We rewrite the integrand as a sum of a polynomial and a proper fraction.

$$\frac{x^4 + 2x^3 + 2x^2 + 10x - 3}{x^3 + 3x} = x + 2 + \frac{-x^2 + 4x - 3}{x(x^2 + 3)}.$$



Now, we decompose the proper fraction into the partial fractions.

$$\frac{-x^2 + 4x - 3}{x(x^2 + 3)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 3}.$$

We clear the fractions by multiplying by a common denominator.

$$-x^2 + 4x - 3 = A(x^2 + 3) + (Bx + C)x.$$

We multiply out the right side of this equality and collect all the like powers of x together.

$$-x^2 + 4x - 3 = (A + B)x^2 + Cx + 3A.$$

We get the following system of equations

$$\begin{array}{rcl} x^2 : & -1 & = A + B, \\ x^1 : & 4 & = C, \\ x^0 : & -3 & = 3A, \end{array}$$

which is easy to solve and we have $A = -1$, $B = 0$ and $C = 4$.



Then the integral is

$$\begin{aligned}\int \frac{x^4 + 2x^3 + 2x^2 + 10x - 3}{x^3 + 3x} dx &= \int x + 2 - \frac{1}{x} + \frac{4}{x^2 + 3} dx = \\ &= \int (x + 2) dx - \int \frac{1}{x} dx + 4 \int \frac{1}{x^2 + 3} dx = \\ &= \frac{x^2}{2} + 2x - \ln |x| + \frac{4}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} + C.\end{aligned}$$



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5. Definite Integral

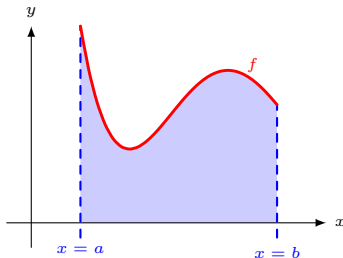
In the last chapter we familiarized us with the integration as a reverse of the derivative. We will be looking at the second type of integral in this chapter: Definite Integral. We will show its applications in geometry and physics.



5.1 Motivation

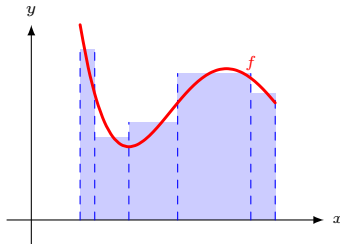
A basic motivation and one of the interpretations of a definite integral, which leads to the definition of the definite integral, is the “area problem”. We want to find the area of a given region in the plane.

For simplicity, let us consider function f that is continuous and positive on a closed interval $[a, b]$. We want to find the area of a region bounded above by the graph of a function f , bounded below by the x -axis, bounded to the left by the vertical line $x = a$ and to the right by the vertical line $x = b$.





There is no simple formula for calculating areas of a general shape. At this point, we can only estimate the area of such regions. Because the area of a rectangle is easy to calculate we can try to approximate the region under the graph of f by rectangles.



If we chose suitable rectangles, the error of an approximation would be small. We can take the height of each rectangle as the value of the function f at the left or right end points of the base intervals of the rectangles. Another estimate can be obtained by using rectangles whose heights are the values of f at the selected points of each base interval.

By taking narrower and narrower rectangles, the resulting approximated areas converge to a certain number, which represents the area of the considered region.



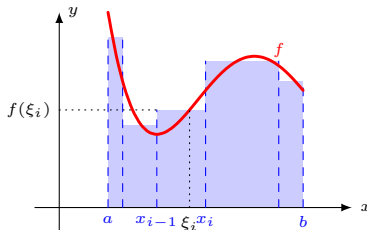
Definition 5.1 (“Math Tutor”, 2019)

Let us consider a closed interval $[a, b]$. By a **partition** of $[a, b]$ we mean any finite set $D = \{x_0, x_1, \dots, x_n\}$ of the points from $[a, b]$ such that $a = x_0 < x_1 < \dots < x_n = b$.

Given partition split the interval $[a, b]$ into n subintervals. In each subinterval

$$[x_0, x_1], [x_1, x_2], \dots, [x_i, x_{i+1}], \dots, [x_{n-1}, x_n]$$

we choose a point $\xi_1, \dots, \xi_i, \dots, \xi_n$. The values $f(\xi_i)$ will define the heights of the rectangles over each subinterval.





The area of the region under the graph of f is approximately

$$A \approx f(\xi_1)(x_1 - x_0) + f(\xi_2)(x_2 - x_1) + \cdots + f(\xi_i)(x_i - x_{i-1}) + \cdots + f(\xi_n)(x_n - x_{n-1}).$$

Definition 5.2 (Mařík, 2012)

Let $[a, b]$ be a closed interval and f be a function defined and bounded on $[a, b]$. Let D be a partition of the interval $[a, b]$. Let $\Xi = \{\xi_1, \dots, \xi_n\}$ be a finite sequence of the points from the interval $[a, b]$ satisfying $x_{i-1} \leq \xi_i \leq x_i$ for $i = 1, \dots, n$. The sum

$$S(f, D, \Xi) = \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1})$$

is said to be an **integral sum** of the function f associated to the partition D and the choice of the numbers x_i in D .

We define the **norm of a partition** D , written $\nu(D)$, to be the largest of all the subinterval widths. As the norm of the partitions of $[a, b]$ approaches zero, the values of the corresponding integral sums approach a limiting value.



Definition 5.3 (Mařík, 2012)

Let $[a, b]$ be a closed interval and f be a function defined and bounded on the interval $[a, b]$. Let D_n be a sequence of partitions of the interval $[a, b]$ which satisfies $\nu(D) \rightarrow 0$ for $n \rightarrow \infty$ and Ξ_n be a sequence of the corresponding choices of numbers ξ from this interval. The function f is said to **be integrable in the sense of Riemann on the interval** $[a, b]$ if there exists a real number $I \in \mathbb{R}$ with property

$$\lim_{n \rightarrow \infty} S(f, D_n, \Xi_n) = I$$

for every sequence D_n , which satisfies

$$\lim_{n \rightarrow \infty} \nu(D_n) = 0$$

and for arbitrary particular choice of the points ξ in Ξ_n . The number I is said to be a **Riemann integral of the function f on the interval** $[a, b]$. We write

$$I = \int_a^b f(x) \, dx.$$



When the condition in the definition is satisfied, we say the Riemann sums of f on $[a, b]$ converge to the definite integral I and that f is integrable over $[a, b]$. We usually say **definite integral** instead of Riemann integral.

The symbol

$$\int_a^b f(x) \, dx$$

is read as “the integral from a to b of f of x dee x ” or “the integral from a to b of f of x with respect to x ”. The symbol \int is called the **integral sign**, it is in fact an elongated S (for sum). The integrated function $f(x)$ is called the **integrand**.

The number a that is at the bottom of the integral sign is called the **lower limit** of the integral and the number b at the top of the integral sign is called the **upper limit** of the integral. The variable of integration is called a dummy variable, since it does not matter what letter we use

$$\int_0^\pi \sin x \, dx = \int_0^\pi \sin u \, du.$$

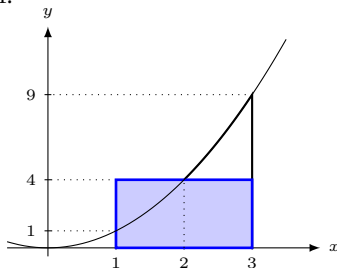
The differential dx indicates that the integral is to be taken with respect to x .



Example 5.4

Estimate the area of the region between $f(x) = x^2$ and the x -axis on $[1, 3]$. Choose $n = 1, 2, 3$ and use the midpoints of the subintervals for the height of the rectangles.

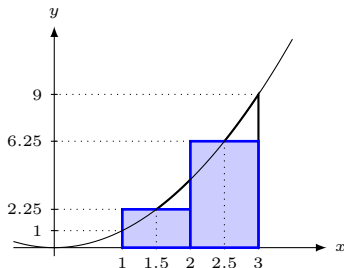
Solution If $n = 1$, then we have the partition $D_1 = \{1, 3\}$ and we choose $\xi_1 = 2$. Considered region is approximated by a rectangle with the interval $[1, 3]$ as the base. Its length equals 2. The second dimension of this rectangle is $f(\xi_0) = f(2) = 4$.



$$\begin{aligned} S_1(f, D) &= \sum_{i=1}^1 f(\xi_i) \cdot (x_i - x_{i-1}) \\ &= 4 \cdot 2 = 8. \end{aligned}$$



If $n = 2$, the interval $[1, 3]$ is split into two subintervals. We have the partition $D_2 = \{1, 2, 3\}$ and we choose $\xi_1 = 1.5$, $\xi_2 = 2.5$. The areas of individual rectangles are $1 \cdot 2.25 = 2.25$ and $1 \cdot 6.25 = 6.25$.

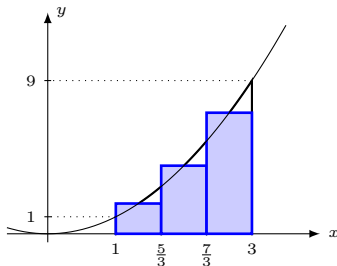


The area of the region between the graph of f and the x -axis is approximately

$$S_2(f, D) = \sum_{i=1}^2 f(\xi_i) \cdot (x_i - x_{i-1}) = 2.25 \cdot 1 + 6.25 \cdot 1 = 8.5.$$



For $n = 3$ we have the partition $D_3 = \{1, \frac{5}{3}, \frac{7}{3}, 3\}$ and $\xi_1 = \frac{4}{3}, \xi_2 = 2, \xi_3 = \frac{8}{3}$.



The area of the region between the graph of f and the x -axis is approximately

$$S_3(f, D) = \sum_{i=1}^3 f(\xi_i) \cdot (x_i - x_{i-1}) = \left(\frac{4}{3}\right)^2 \cdot \frac{2}{3} + (2)^2 \cdot \frac{2}{3} + \left(\frac{8}{3}\right)^2 \cdot \frac{2}{3} \doteq 8.59259.$$

The exact value of the area is $S = \frac{26}{3} \doteq 8.6667$.



The lower limit a of the integral and the upper limit b represent the **interval of integration** $[a, b]$. Usually we put the left end point as a lower limit. But we can integrate with any order of limits because for $a > b$ it is defined

$$\int_b^a f(x) \, dx = - \int_a^b f(x) \, dx.$$

It can be also proved

$$\int_a^a f(x) \, dx = 0.$$

Theorem 5.5 (Existence of the Riemann Integral) (“Math Tutor”, 2019)

Let a function f be continuous or monotonous in an interval $[a, b]$. Then $\int_a^b f(x) \, dx$ exists.

Not only continuous functions but also the functions with a finite number of jump discontinuities, so called piecewise-continuous functions, are integrable on a closed interval.



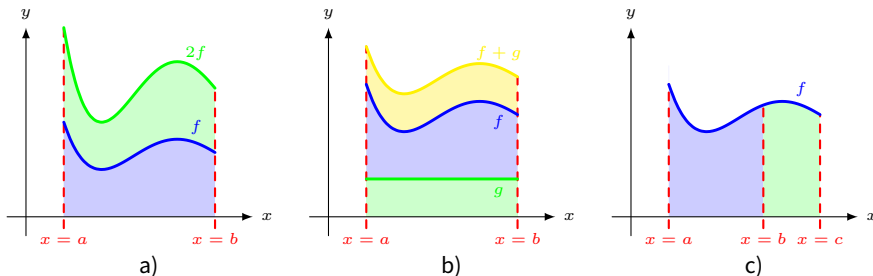
5.2 Properties of the Riemann Integral

Let f and g be integrable on the interval $[a, b]$ and c be a real number. The definite integral has the following properties.

- a) $\int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx.$
- b) $\int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx.$
- c) $\int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx.$
- d) $f(x) \leq g(x) \text{ on } (a, b) \Rightarrow \int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx.$
- e) If f is an odd function, then $\int_{-a}^a f(x) \, dx = 0.$
- f) If f is an even function, then $\int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx.$



Let us sketch geometric interpretations of some properties. We use the fact that the Riemann integral is understood as the area of some region.



Constant multiple and a sum of integrals are shown in the figures a), b). The figure c) depicts additivity of the definite integrals.

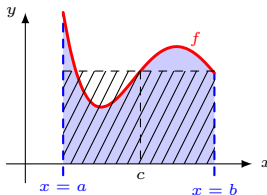


Theorem 5.6 (The Mean Value Theorem for Integrals) (“Math Tutor”, 2019)

Let f be a continuous function on $[a, b]$. Then there is a number $c \in (a, b)$ such

$$\int_a^b f(x) \, dx = f(c) \cdot (b - a).$$

This theorem is known as the **First Mean Value Theorem for Integrals**. The point $f(c)$ is called the **average value of $f(x)$ on $[a, b]$** . From a geometric point of view the area of the region under the graph of f equals to the area of the rectangle with the height equalled to $f(c)$ and the basis $[a, b]$.





5.3 The Fundamental Theorem of Calculus

In this section we will show the connection between the Riemann definite integral and antiderivatives.

Theorem 5.7 (“Math Tutor”, 2019)

Let f be a function that is Riemann integrable on $[a, b]$, let c belong to $[a, b]$. For $x \in [a, b]$ define

$$F(x) = \int_c^x f(t) \, dt.$$

Then F is a continuous function on $[a, b]$. Moreover, for $x \in (a, b)$, if f is continuous at x , then F is differentiable at x and $F'(x) = f(x)$.

This is the first part of the Fundamental Theorem of Calculus. It enables us to compute an antiderivative of the integrand using definite integrals.

The second part of the Fundamental Theorem of Calculus is also called the **Newton-Leibniz Formula**. Instead of using a Riemann sum to find definite integral we evaluate an antiderivative at the upper and lower limits of integration.



Theorem 5.8 (The Newton-Leibniz Formula) (Bouchala & Sadowská, 2007)

Let f be a continuous function on $[a, b]$. If F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a).$$

To evaluate a definite integral of f over an interval $[a, b]$ we find an antiderivative F of f and calculate the number $F(b) - F(a)$. The antiderivative $F(x) + C$ can be chosen arbitrary, so we use $C = 0$.

Example 5.9

Evaluate the integral $\int_0^2 x^2 + 2x dx$.

Solution

$$\int_0^2 x^2 + 2x dx = \left[\frac{x^3}{3} + x^2 \right]_0^2 = \frac{2^3}{3} + 2^2 - \left(\frac{0^3}{3} + 0^2 \right) = \frac{8}{3} + 4 = \frac{20}{3}.$$



5.4 Integration by Parts for Definite Integrals

We have two ways to integrate the definite integral $\int_a^b u(x)v'(x) \, dx$ by parts.

1. Evaluate the indefinite integral

$$I(x) = \int u(x)v'(x) \, dx,$$

then

$$\int_a^b u(x)v'(x) \, dx = [I(x)]_a^b = I(b) - I(a).$$

2. Use the formula

$$\int_a^b u'(x)v(x) \, dx = [u(x)v(x)]_a^b - \int_a^b u(x)v'(x) \, dx.$$

Example 5.10

Evaluate the integral $\int_0^\pi x \sin x \, dx$.



Solution The integration by parts formula gives

$$\begin{aligned}\int_0^{\pi} x \sin x \, dx &= \left| \begin{array}{ll} u = x & v' = \sin x \\ u' = 1 & v = -\cos x \end{array} \right| = [-x \cos x]_0^{\pi} + \int_0^{\pi} \cos x \, dx = \\ &= -\pi \cdot \cos \pi + 0 \cdot \cos 0 + [\sin x]_0^{\pi} = \pi + \sin \pi - \sin 0 = \pi.\end{aligned}$$

Example 5.11

Evaluate the integral $\int_1^e x \ln x \, dx$.

Solution Integrating by parts we have

$$\begin{aligned}\int_1^e x \ln x \, dx &= \left| \begin{array}{ll} u = \ln x & v' = x \\ u' = 1/x & v = x^2/2 \end{array} \right| = \left[\frac{x^2}{2} \ln x \right]_1^e - \int_1^e \frac{1}{x} \cdot \frac{x^2}{2} \, dx = \\ &= \frac{e^2}{2} \ln e - \frac{1^2}{2} \ln 1 - \frac{1}{2} \int_1^e x \, dx = \frac{e^2}{2} - \frac{1}{2} \left[\frac{x^2}{2} \right]_1^e = \frac{e^2}{2} - \frac{1}{4} (e^2 - 1) = \frac{e^2}{4} + \frac{1}{4}.\end{aligned}$$



5.5 Substitution Rule for Definite Integrals

There are two ways how to evaluate the definite integral $\int_a^b f(g(x)) g'(x) dx$ using substitution.

1. The first step is to compute the indefinite integral using substitution. Then, we can apply the Newton-Leibniz Formula to evaluate the definite integral.
2. Using substitution $t = g(x)$ we eliminate all the x 's in the integral including the limits of the integral. It means

$$\int_a^b f(g(x)) g'(x) dx = \int_{g(a)}^{g(b)} f(t) dt.$$

If we convert the integral and also its limits to t 's terms, we do not return back to the original variable.

Example 5.12

Evaluate the integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^3 x \cos x dx$.



Solution Let us put $t = \sin x$, $dt = \cos x \, dx$. We need to change the limits of the integral. If $x = \frac{\pi}{4}$ then $t = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. For $x = \frac{\pi}{2}$ we have $t = \sin\left(\frac{\pi}{2}\right) = 1$.

$$\begin{aligned} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^3 x \cos x \, dx &= \left| \begin{array}{l} t = \sin x \\ dt = \cos x \, dx \\ x = \frac{\pi}{4} \rightarrow t = \frac{\sqrt{2}}{2} \\ x = \frac{\pi}{2} \rightarrow t = 1 \end{array} \right| = \int_{\frac{\sqrt{2}}{2}}^1 t^3 \, dt = \\ &= \left[\frac{t^4}{4} \right]_{\frac{\sqrt{2}}{2}}^1 = \frac{1^4}{4} - \frac{\left(\frac{\sqrt{2}}{2}\right)^4}{4} = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}. \end{aligned}$$

Example 5.13

Evaluate the integral $\int_2^3 \frac{1}{(2x-1)^3} \, dx$.

Solution Let us consider $t = 2x - 1$.



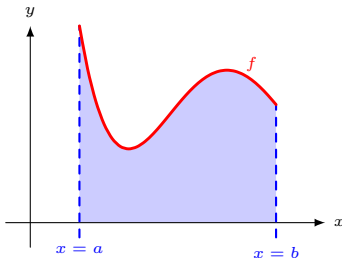
The integral is

$$\begin{aligned} \int_2^3 \frac{1}{(2x-1)^3} dx &= \left| \begin{array}{l} t = 2x - 1 \\ dt = 2 dx \\ \frac{1}{2} dt = dx \\ x = 2 \rightarrow t = 3 \\ x = 3 \rightarrow t = 5 \end{array} \right| = \frac{1}{2} \int_3^5 \frac{1}{t^3} dt = \frac{1}{2} \int_3^5 t^{-3} dt = \\ &= \frac{1}{2} \left[\frac{t^{-2}}{-2} \right]_3^5 = -\frac{1}{4} \left[\frac{1}{t^2} \right]_3^5 = -\frac{1}{4} \cdot \left(\frac{1}{25} - \frac{1}{9} \right) = \frac{4}{225}. \end{aligned}$$



5.6 Area of a Planar Region

Let f be a continuous and positive function on a closed interval $[a, b]$. Let us consider a region bounded above by the graph of a function f , bounded below by the x -axis, bounded to the left by the vertical line $x = a$ and to the right by the vertical line $x = b$.

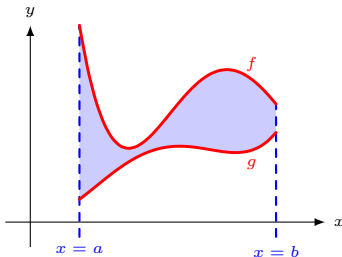


The area of such region is

$$A = \int_a^b f(x) \, dx.$$



Let f and g be continuous functions with $f(x) \geq g(x)$ for $x \in [a, b]$. Let us consider the region bounded from above by the graph of a function f and from below by the graph of a function g on an interval $[a, b]$.



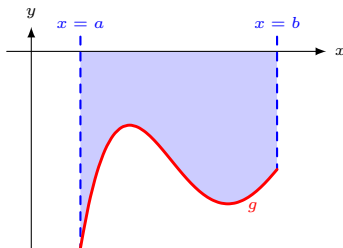
The area of this region is

$$A = \int_a^b f(x) - g(x) \, dx.$$

It is useful to graph the curves. Their graphs will show which curve is the upper curve f and which is the lower curve g , or where the curves intersect to determine limits of integration.



If $f = 0$, $g < 0$ on $[a, b]$, the integral $\int_a^b f(x) \, dx$ is negative.

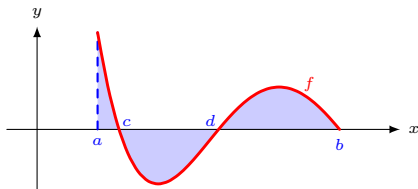


The area of the region bounded from above by the x -axis and bounded from below by the graph of a function g is

$$A = \left| \int_a^b g(x) \, dx \right| = - \int_a^b g(x) \, dx.$$



Let us consider the following region.

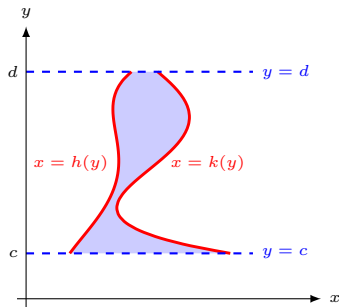


The function f changes its sign on $[a, b]$. We split the interval $[a, b]$ into subintervals $[a, c]$, $[c, d]$, $[d, b]$. The area of considered region is the sum of areas of the subregions.

$$A = \int_a^c f(x) \, dx - \int_c^d f(x) \, dx + \int_d^b f(x) \, dx.$$



Let us consider a region bounded to the left by the graph of a function $x = h(y)$, to the right by the graph of a function $x = k(y)$, bounded above by the horizontal line $y = d$ and bounded below by the horizontal line $y = c$.



The area of this region is

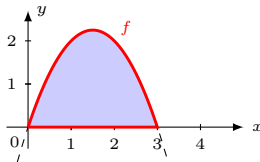
$$A = \int_c^d k(y) - h(y) \, dy.$$



Example 5.14

Determine the area of the region bounded by the graph of $f(x) = 3x - x^2$ and the x -axis.

Solution Let us sketch the graph of f .



The graph of f intersects with the x -axis in two points,

$$3x - x^2 = 0 \quad \Rightarrow \quad x = 0, x = 3,$$

which are the limits of integration. The area is

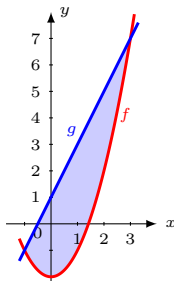
$$A = \int_a^b f(x) \, dx = \int_0^3 (3x - x^2) \, dx = \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 = \frac{27}{2} - \frac{27}{3} = \frac{9}{2}.$$



Example 5.15

Determine the area of the region bounded by the graphs of $f(x) = x^2 - 2$ and $g(x) = 2x + 1$.

Solution First we sketch both of graphs.



The limits of integration are the intersection points of f with g . They can be found by setting the two functions equal.



$$x^2 - 2 = 2x + 1$$

$$x^2 - 2x - 3 = 0$$

$$x = -1, \quad x = 3.$$

The region starts from $x = -1$ to $x = 3$. The limits of integration are $a = -1$, $b = 3$. The area between the curves is

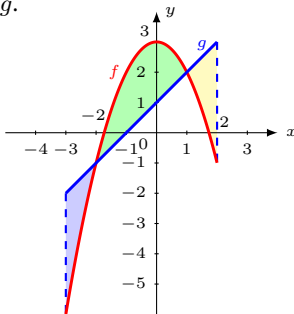
$$\begin{aligned} A &= \int_a^b ((\text{upper function}) - (\text{lower function})) \, dx = \\ &= \int_{-1}^3 (2x + 1 - (x^2 - 2)) \, dx = \int_{-1}^3 (2x + 3 - x^2) \, dx = \\ &= \left[x^2 + 3x - \frac{x^3}{3} \right]_{-1}^3 = 3^2 + 3 \cdot 3 - \frac{3^3}{3} - \left((-1)^2 + 3 \cdot (-1) - \frac{(-1)^3}{3} \right) = \\ &= 9 - \left(-\frac{5}{3} \right) = \frac{32}{3}. \end{aligned}$$



Example 5.16

Determine the area of the region bounded by the graphs of $f(x) = -x^2 + 3$, $g(x) = x + 1$ for $x \in [-3, 2]$.

Solution Let us sketch the graphs of f and g .



The bounding curves change at two points, we subdivide the region into three subregions, calculate area of each subregion between curves and then add them up.



The graph of f intersects with the graph of g in two points

$$-x^2 + 3 = x + 1$$

$$x^2 + x - 2 = 0$$

$$x = -2, \quad x = 1.$$

There are three intervals $[-3, -2]$, $[-2, 1]$, $[1, 2]$ on which one function is always the upper function and the other is always the lower function. The total area is a sum of three integrals

$$\begin{aligned} A &= \int_{-3}^{-2} (g(x) - f(x)) \, dx + \int_{-2}^1 (f(x) - g(x)) \, dx + \int_1^2 (g(x) - f(x)) \, dx = \\ &= \int_{-3}^{-2} (x + 1 - (-x^2 + 3)) \, dx + \int_{-2}^1 (-x^2 + 3 - (x + 1)) \, dx + \int_1^2 (x + 1 - (-x^2 + 3)) \, dx = \\ &= \int_{-3}^{-2} (x^2 + x - 2) \, dx + \int_{-2}^1 (-x^2 - x + 2) \, dx + \int_1^2 (x^2 + x - 2) \, dx. \end{aligned}$$



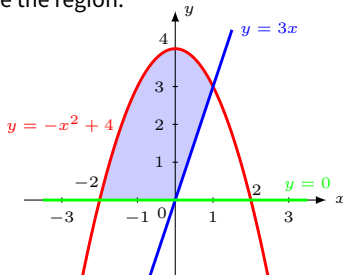
$$\begin{aligned} A &= \left[\frac{x^3}{3} + \frac{x^2}{2} - 2x \right]_{-3}^{-2} + \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^1 + \left[\frac{x^3}{3} + \frac{x^2}{2} - 2x \right]_1^2 \\ &= \left[\frac{(-2)^3}{3} + \frac{(-2)^2}{2} - 2(-2) - \left(\frac{(-3)^3}{3} + \frac{(-3)^2}{2} - 2(-3) \right) \right] + \\ &\quad + \left[-\frac{1^3}{3} - \frac{1^2}{2} + 2 \cdot 1 - \left(-\frac{(-2)^3}{3} - \frac{(-2)^2}{2} + 2(-2) \right) \right] + \\ &\quad + \left[\frac{2^3}{3} + \frac{2^2}{2} - 2 \cdot 2 - \left(\frac{1^3}{3} + \frac{1^2}{2} - 2 \cdot 1 \right) \right] \\ &= \frac{11}{6} + \frac{9}{2} + \frac{11}{6} = \frac{49}{6}. \end{aligned}$$



Example 5.17

Determine the area of the region defined by the inequalities $y \leq -x^2 + 4$, $y \geq 3x$, $y \geq 0$.

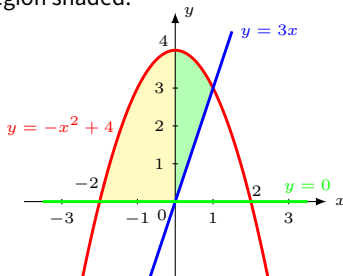
Solution First we graph the three curves to see the region.



The region we are interested in must have one of the three curves on every boundary of the region. The region's upper boundary is the graph of $y = -x^2 + 4$. The lower boundary is formed by two parts. We have to divide the region into two parts.



Here is a picture of the region with each subregion shaded.



The points of intersection of the three graphs can be identified from the picture. Of course, we can find them also by calculation.

$$\begin{aligned} -x^2 + 4 &= 3x \\ x^2 + 3x - 4 &= 0 \\ x &= -4, \quad x = 1 \end{aligned}$$

$$\begin{aligned} -x^2 + 4 &= 0 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

$$\begin{aligned} 3x &= 0 \\ x &= 0 \end{aligned}$$



The limits of integration for yellow region are $a = -2$, $b = 0$. The limits of integration for green region are $a = 0$, $b = 1$. The total area is

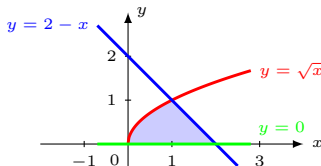
$$\begin{aligned} A &= \int_{-2}^0 (-x^2 + 4) \, dx + \int_0^1 (-x^2 + 4 - 3x) \, dx = \\ &= \left[-\frac{x^3}{3} + 4x \right]_{-2}^0 + \left[-\frac{x^3}{3} + 4x - \frac{3x^2}{2} \right]_0^1 = \\ &= 0 - \left(-\frac{(-2)^3}{3} + 4 \cdot (-2) \right) - \frac{1^3}{3} + 4 \cdot 1 - \frac{3 \cdot 1^2}{2} - 0 = \\ &= -\frac{8}{3} + 8 - \frac{1}{3} + 4 - \frac{3}{2} = \frac{-16 + 48 - 2 + 24 - 9}{6} = \frac{15}{2}. \end{aligned}$$



Example 5.18

Determine the area of the region bounded by the graphs of $y = \sqrt{x}$, $y = 2 - x$, $y = 0$.

Solution First we graph the three curves to see the region.



The region's lower boundary is the graph of $y = 0$. The upper boundary is formed by two parts. We have to divide the region into two parts. But we will try another way to determine the area. It is necessary to rewrite boundary functions into the form $x = f(y)$:

$$y = \sqrt{x}$$

$$x = y^2$$

$$y = 2 - x$$

$$x = 2 - y.$$



The region's left boundary is the graph of $x = y^2$, the region's right boundary is the graph of $x = 2 - y$. We switch the axes to draw new situation. The point of intersection of the graphs of $x = y^2$ and $x = 2 - y$ can be identified from the picture. We verify this by calculation.

$$\begin{aligned}y^2 &= 2 - y \\y^2 + y - 2 &= 0 \\y &= 1 \quad (y = -2 \text{ not in the region}).\end{aligned}$$

The limits of integration are $c = 0$, $d = 1$.

$$\begin{aligned}A &= \int_c^d ((\text{right function}) - (\text{left function})) \, dy = \\&= \int_0^1 (2 - y - y^2) \, dx = \left[2y - \frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 = \\&= 2 \cdot 1 - \frac{1^2}{2} - \frac{1^3}{3} = \frac{7}{6}.\end{aligned}$$



5.7 Length of a Curve

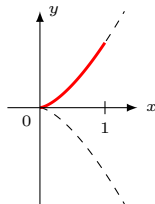
We want to determine the length of the graph of the continuous function $y = f(x)$ on the interval $[a, b]$. If f has a continuous derivative on $[a, b]$, then the length of its graph is

$$L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx.$$

Example 5.19

Determine the length of the curve $f(x) = x^{\frac{3}{2}}$ on $[0, 1]$.

Solution Here is the graph of f .





Let us compute the derivative of f and its square.

$$f'(x) = \left(x^{\frac{3}{2}}\right)' = \frac{3}{2}x^{\frac{1}{2}}, \quad (f'(x))^2 = \frac{9}{4}x.$$

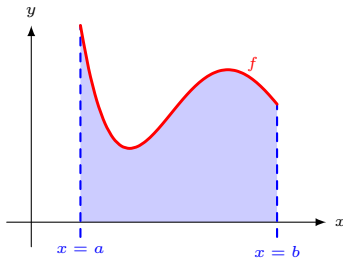
Then the length is

$$\begin{aligned} L &= \int_a^b \sqrt{1 + (f'(x))^2} \, dx = \int_0^1 \sqrt{1 + \frac{9}{4}x} \, dx = \\ &= \left[\begin{array}{l} t = 1 + \frac{9}{4}x \\ dt = \frac{9}{4} \, dx \\ \frac{4}{9} \, dt = dx \\ x = 0 \rightarrow t = 1 \\ x = 1 \rightarrow t = \frac{13}{4} \end{array} \right] = \frac{4}{9} \int_1^{\frac{13}{4}} t^{\frac{1}{2}} \, dx = \frac{4}{9} \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^{\frac{13}{4}} = \\ &= \frac{4}{9} \cdot \frac{2}{3} \cdot \left(\left(\frac{13}{4} \right)^{\frac{3}{2}} - 1 \right) = \frac{8}{27} \cdot \left(\sqrt{\left(\frac{13}{4} \right)^3} - 1 \right) = \frac{8}{27} \cdot \left(\frac{13\sqrt{13}}{8} - 1 \right). \end{aligned}$$

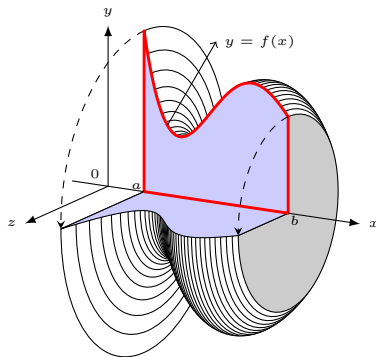


5.8 Volume of a Solid of Revolution

Let f be a continuous and non-negative function on an interval $[a, b]$. Let us consider a region bounded above by the graph of a function f and bounded below by the x -axis on an interval $[a, b]$.



We revolve this region about the x -axis to get the solid of revolution. The region above on the picture gives the following three dimensional region.

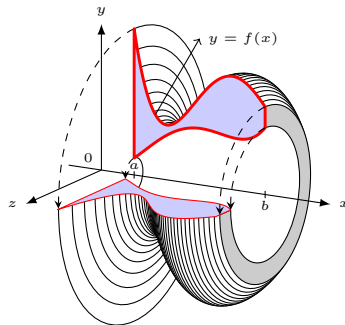
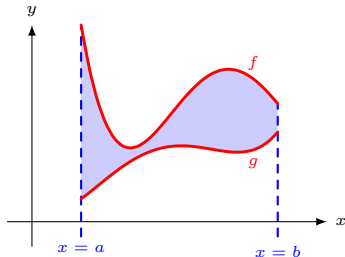


The volume of this solid is

$$V = \pi \int_a^b f^2(x) \, dx.$$



Let f and g be continuous, non-negative functions with $f(x) \geq g(x)$ for $x \in [a, b]$. Let us consider the region bounded from above by the graph of a function f and from below by the graph of a function g on an interval $[a, b]$.



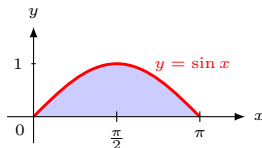
The volume of this solid is
$$V = \pi \int_a^b (f^2(x) - g^2(x)) \, dx.$$



Example 5.20

Determine the volume of the solid obtained by rotating the region bounded by $f(x) = \sin x$ and the x -axis on $[0, \pi]$ about the x -axis.

Solution First we sketch the region.



The volume is

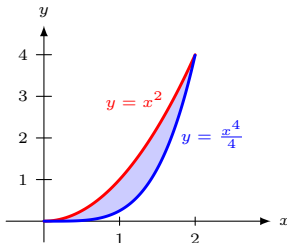
$$\begin{aligned} V &= \pi \int_a^b f^2(x) \, dx = \pi \int_0^\pi (\sin x)^2 \, dx = \pi \int_0^\pi \frac{1 + \cos 2x}{2} \, dx = \\ &= \frac{\pi}{2} \int_0^\pi (1 + \cos 2x) \, dx = \frac{\pi}{2} \cdot \left[x + \frac{\sin 2x}{2} \right]_0^\pi = \frac{\pi}{2} \cdot \left(\pi + \frac{\sin 2\pi}{2} - 0 \right) = \frac{\pi^2}{2}. \end{aligned}$$



Example 5.21

Determine the volume of the solid obtained by rotating the region bounded by the graphs of $y = x^2$ and the $y = \frac{x^4}{4}$.

Solution First we sketch the region.



We need to determine the intersection points.



$$x^2 = \frac{x^4}{4}$$

$$\frac{x^4}{4} - x^2 = 0$$

$$\frac{x^2}{4} \cdot (x^2 - 4) = 0$$

$$x = -2, \quad x = 0, \quad x = 2 \quad \Rightarrow (0, 0), (2, 4).$$

The volume is

$$\begin{aligned} V &= \pi \int_a^b (f^2(x) - g^2(x)) \, dx = \pi \int_0^2 \left((x^2)^2 - \left(\frac{x^4}{4} \right)^2 \right) dx = \\ &= \pi \int_0^2 \left(x^4 - \frac{x^8}{16} \right) dx = \pi \left[\frac{x^5}{5} - \frac{x^9}{144} \right]_0^2 = \pi \left(\frac{32}{5} - \frac{512}{144} \right) = \frac{128}{45} \pi. \end{aligned}$$



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Mathematics I

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Introduction

This paper provides tutorials and explanatory notes for the first course of *Mathematics*. The content is organized by learning objectives and it complements and extends the lecture presentation slides for Mathematics I.



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1 Real Functions

In this chapter, we will study real functions of one real variable. We will determine their domains and show some algebraic techniques useful for working with the functions. We will determine basic properties of the functions and graph the functions using transformations.

1.1 Sets of Reals Numbers

In this section we will give a quick review of “the set theory”, real numbers and some related vocabulary.

By a set we usually mean a collection of some objects, which are called the **elements of the set**. We use upper-case letters to denote the sets and lower-case letters for their elements. If an element a belongs to a set A , we denote it by $a \in A$. If an element b does not belong to a set A , we denote it $b \notin A$.

We can describe a set in the followings ways:

- a list of each element of the set in braces $(\{, \})$: $M = \{s, e, t\}$,
- a set-builder notation with a variable and a logical predicate:

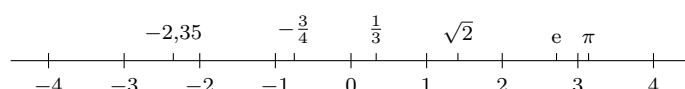
$$M = \{x : \text{is a letter in the word “set”}\}.$$

The way to read this is: “The set of elements x such that x is a letter in the word “set””.

The important sets are the sets of numbers:

$\mathbb{N} = \{1, 2, 3, \dots\}$	the set of the natural numbers,
$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$	the set of the whole numbers,
$\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\}$	the set of the rational numbers,
\mathbb{R}	the set of the real numbers,
$\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$	the set of the irrational numbers,
\mathbb{C}	the set of the complex numbers.

The real numbers can be represented as the points on a number line called **the real line**.



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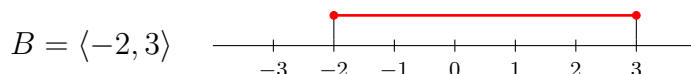
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Segments of this line are called **intervals** of numbers. Intervals can be represented either directly on the number line or by horizontal lines parallel to the number line. Empty circle marks the point that does not belong to the interval, the full circle indicates that the point belongs to the interval.

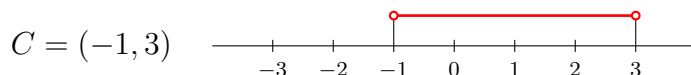
Let $a, b \in \mathbb{R}$, $a < b$. **Closed interval** with the endpoints a and b is the set

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$



Open interval with the endpoints a and b is the set

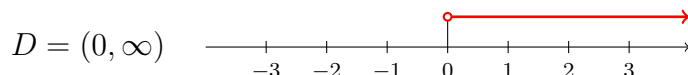
$$(a, b) = \{x \in \mathbb{R} : a < x < b\}.$$



Half-open interval with the endpoints a and b is the set $[a, b)$ or $(a, b]$.

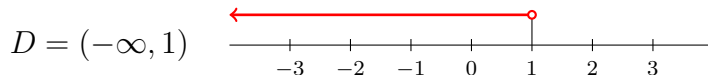
Open interval (a, ∞) is the set

$$\{x \in \mathbb{R} : a < x\}.$$



Open interval $(-\infty, b)$ is the set

$$\{x \in \mathbb{R} : x < b\}.$$



There are similar definitions for the other intervals $[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$ and $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$.

For two sets A and B we define the following operations for constructing new sets from given sets.

Operation	Symbol	Definition
The union of A and B	$A \cup B$	The set of all things that are members of either A or B .
The intersection of A and B	$A \cap B$	The set of all things that are members of both A and B .
The difference of A and B	$A \setminus B$	The set of all things that are members of A but not members of B .
The complement of A in a given universal set U	A'_U	The set of all things that are members of U but not members of A .

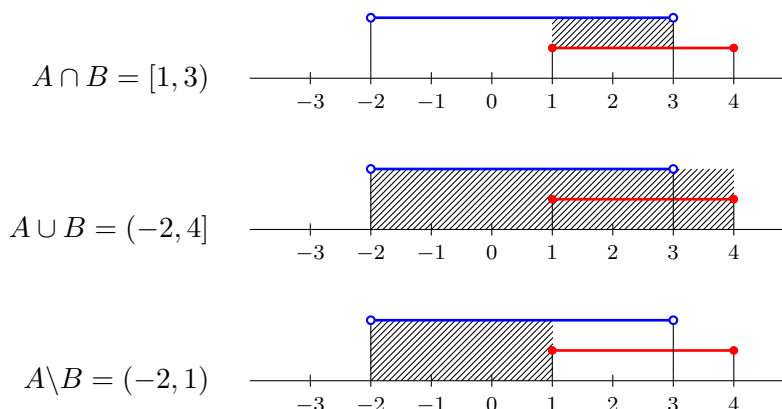
If $A \cap B = \emptyset$, we say that A and B are disjoint.





Example 1. For the intervals $A = (-2, 3)$ and $B = [1, 4]$ find their $A \cap B$, $A \cup B$, $A \setminus B$.

Solution: To find $A \cap B$ we shade the overlap of the two intervals and obtain $A \cap B = [1, 3)$. To find $A \cup B$ we shade each of A and B and the result is $A \cup B = (-2, 4]$. And finally, to find $A \setminus B$ we shade that part of A which does not contain elements of B , so $A \setminus B = (-2, 1)$.



□

1.2 Definition of a Function

1.2.1 Mappings

Precise definition of a real function of real variable requires the knowledge of the *mappings*. Let us consider sets A and B . Their **Cartesian product** is defined as the set

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}.$$

By a **mapping** from A to B we mean any subset T of the Cartesian product $A \times B$ that satisfies the following condition:

For every $a \in A$ there is a unique $b \in B$ such that $(a, b) \in T$.

We write $T : A \mapsto B$ (MT). By a **real function of real variable** we mean any mapping from some subset of the set of real numbers to the set of real numbers.

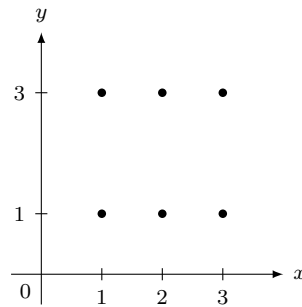
Example 2. Consider sets $A = \{1, 2, 3\}$, $B = \{1, 3\}$. Determine and draw their Cartesian product.

Solution: The Cartesian product $A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$:

$$A \times B = \{(1, 1), (1, 3), (2, 1), (2, 3), (3, 1), (3, 3)\}.$$

This Cartesian product consists of the ordered pairs of the real numbers. The first number in the ordered pair is called the **abscissa** or **x -coordinate** and the second is called the **ordinate** or **y -coordinate**. The ordered pair forms the **Cartesian coordinates** of some point. We can draw the Cartesian product $A \times B$ like this





□

Example 3. Determine which of the following sets is a function and which is not a function.

- a) $T_1 = \{(-2, 0), (-1, 0), (2, 1), (3, 4), (4, 6)\}$,
 b) $T_2 = \{(-2, 0), (-1, 1), (-1, 0), (1, 3)\}$.

Solution: a) Let us denote the set of the first numbers from each ordered pair by A and the set of the second numbers from each ordered pair by B

$$A = \{-2, -1, 2, 3, 4\}, \quad B = \{0, 1, 4, 6\}.$$

To see whether the set T_1 is a function, there must be exactly one value from B associated with any value from the set A . For example if we check -2 from the set of A , there is exactly one ordered pair with -2 as a first component, $(-2, 0)$. So there is only one value from B associated with -2 , number 0. Note that 0 is the second component of the second ordered pair in T_1 . This is no problem, the important thing is that there is only one ordered pair with -2 as a first component.

b) We can see that there are two ordered pairs with -1 as a first component: $(-1, 1)$, $(-1, 0)$. It means there are two values from B associated with -1 . The set T_2 is not a function.

□

Example 4. Determine whether the relation $y = 3x + 2$ is a function.

Solution: Definition 1.2 (See the lecture presentation slides.) says that a function $f : A \rightarrow B$ is a rule which associates each element x of the set A with exactly one element y of the set B . The equation $y = 3x + 2$ will define a function if for all possible values x plugged into the equation we will get exactly one value y . If we multiply a particular number by 3 we will get a single value. We will also get only a single value if we add 2 to a number. Based on these operations we can say that after plugging an arbitrary x into the equation $y = 3x + 2$ we will get exactly one value y . So, the equation $y = 3x + 2$ is a function.

□

1.2.2 Domain of a Function

A **domain** of the function f is the set of all numbers x that can be substituted into the function, that is, for which the formula defining the function makes sense. It is denoted by $D(f)$ (or D_f).



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Example 5. Determine the domain of the function $f(x) = \frac{x-2}{x^2-x-12}$.

Solution: The domain for this function are all the values x for which we do not have division by zero. We need set the denominator equal to zero and solve.

$$x^2 - x - 12 = 0 \Rightarrow x_{1,2} = \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-12)}}{2} = \frac{1 \pm 7}{2} \Rightarrow x_1 = -3, x_2 = 4.$$

We will get division by zero if we plug in $x = -3$ or $x = 4$. So, the domain is $D_f = (-\infty, -3) \cup (-3, 4) \cup (4, \infty)$. □

Example 6. Determine the domain of the function $f(x) = \sqrt{x^2 - 3x - 4}$.

Solution: We have a square root, so the expression under the root is required to be non-negative. The condition is

$$x^2 - 3x - 4 \geq 0.$$

We need to determine where the polynomial is zero. If possible, we can factor the polynomial or use the quadratic formula with the discriminant.

$$x^2 - 3x - 4 = 0 \Rightarrow (x-4)(x+1) = 0 \Rightarrow x_1 = -1, x_2 = 4.$$

The polynomial is zero at $x = -1$ and $x = 4$. These points divide the number line into three intervals. We will pick test points from each region to see if $x^2 - 3x - 4$ is positive or negative in each region. We will determine its sign by evaluating $x^2 - 3x - 4$ at a convenient point in each interval.

$$\begin{array}{ccccccc} & + & & - & & + & \\ & | & & | & & & \\ \hline & -1 & & 4 & & & \end{array}$$

We can see that $x^2 - 3x - 4$ is positive for $x < -1$ and $x > 4$. Hence, the domain is $D_f = (-\infty, -1] \cup [4, \infty)$. □

Example 7. Determine the domain of the function $f(x) = \frac{\sqrt{3x-4}}{x^2-9}$.

Solution: We have both a square root and division by zero. The conditions are

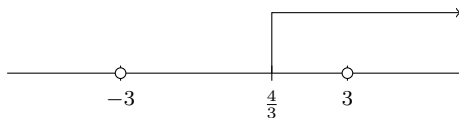
$$3x - 4 \geq 0 \quad \text{and} \quad x^2 - 9 \neq 0.$$

It is required that

$$\begin{array}{ll} 3x - 4 \geq 0 & \wedge \quad x^2 - 9 \neq 0 \\ 3x \geq 4 & \wedge \quad x^2 \neq 9 \\ x \geq \frac{4}{3} & \wedge \quad x \neq \pm 3. \end{array}$$

We can draw this using the number line.





The function f is defined for all $x \geq \frac{4}{3}$ except $x = 3$, $D_f = (\frac{4}{3}, 3) \cup (3, \infty)$.

□

Example 8. Determine the domain of the function $f(x) = 3 \ln(x^2 - 10x + 21)$.

Solution: The domain of a logarithmic function is $(0, \infty)$, so the argument of a logarithm is required to be positive. We set

$$x^2 - 10x + 21 > 0.$$

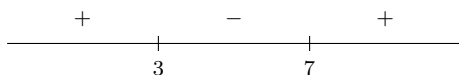
We find the zeros of $x^2 - 10x + 21$ by solving $x^2 - 10x + 21 = 0$ for x . Factoring gives

$$x^2 - 10x + 21 = 0 \Rightarrow (x - 3)(x - 7) = 0 \Rightarrow x_1 = 3, x_2 = 7.$$

These points divide the number line into three intervals

$$(-\infty, 3), (3, 7), (7, \infty).$$

We will pick test points from each interval to see if $x^2 - 10x + 21$ is positive or negative in each interval.



We can see that $x^2 - 10x + 21$ is positive for $x < 3$ and $x > 7$. Hence, the domain is $D_f = (-\infty, 3) \cup (7, \infty)$.

□

Example 9. Determine the domain of the function $f(x) = \arcsin \frac{2x - 5}{4}$.

Solution: The arcsine function is defined on $[-1, 1]$, thus we need to set

$$-1 \leq \frac{2x - 5}{4} \leq 1.$$

These two inequalities have to be satisfied simultaneously, but we can solve them both simultaneously and individually.

$$\begin{aligned} -1 &\leq \frac{2x - 5}{4} & \wedge & \quad \frac{2x - 5}{4} \leq 1 \\ -4 &\leq 2x - 5 & \wedge & \quad 2x - 5 \leq 4 \\ 1 &\leq 2x & \wedge & \quad 2x \leq 9 \\ \frac{1}{2} &\leq x & \wedge & \quad x \leq \frac{9}{2}. \end{aligned}$$

We obtain the domain as $D_f = [\frac{1}{2}, \frac{9}{2}]$.

□





Example 10. Determine the domain of the function $f(x) = \arccos \frac{1-2x}{3} + \ln(6x-3)$.

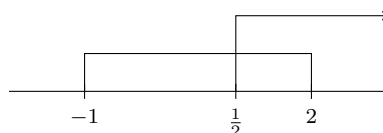
Solution: The arccosine function is defined on $[-1, 1]$ and the argument of a logarithm must be positive, thus we set

$$-1 \leq \frac{1-2x}{3} \leq 1 \quad \wedge \quad 6x-3 > 0.$$

At first, we solve the inequalities for the arccosine function.

$$\begin{aligned} -1 &\leq \frac{1-2x}{3} & \wedge & \quad \frac{1-2x}{3} \leq 1 \\ -3 &\leq 1-2x & \wedge & \quad 1-2x \leq 3 \\ -4 &\leq -2x & \wedge & \quad -2x \leq 2 \\ 2 &\geq x & \wedge & \quad x \geq -1. \end{aligned}$$

The next condition $6x-3 > 0$ leads to inequality $x > \frac{1}{2}$. We put all these conditions together graphically by the number line.



We obtain the domain as $D_f = (\frac{1}{2}, 2]$.

□

1.3 Operations with Functions

We will use basic algebraic operations with functions to solve the following examples.

Example 11. For given function $f(x) = 2x^2 - 1$ find $f(2)$, $f(-1)$, $f(x-2)$, $f(x^2)$, $f(2a)$, $2f(a)$, $f(a) + f(3)$, $f(a+3)$, $f(a+h)$, $\frac{f(a+h)-f(a)}{h}$.

Solution:

- $f(2) = 2 \cdot 2^2 - 1 = 7$,
- $f(-1) = 2 \cdot (-1)^2 - 1 = 1$,
- $f(x-2) = 2 \cdot (x-2)^2 - 1 = 2(x^2 - 4x + 4) - 1 = 2x^2 - 8x + 7$,
- $f(x^2) = 2 \cdot (x^2)^2 - 1 = 2x^4 - 1$,
- $f(2a) = 2 \cdot (2a)^2 - 1 = 8a^2 - 1$,
- $2f(a) = 2 \cdot (2a^2 - 1) = 4a^2 - 2$,





- $f(a) + f(3) = 2 \cdot a^2 - 1 + 2 \cdot 3^2 - 1 = 2a^2 + 16,$
- $f(a + 3) = 2 \cdot (a + 3)^2 - 1 = 2(a^2 + 6a + 9) - 1 = 2a^2 + 12a + 17,$
- $f(a + h) = 2 \cdot (a + h)^2 - 1 = 2(a^2 + 2ah + h^2) - 1 = 2a^2 + 4ah + 2h^2 - 1,$
- $\frac{f(a + h) - f(a)}{h} = \frac{2a^2 + 4ah + 2h^2 - 1 - (2a^2 - 1)}{h} = \frac{4ah + 2h^2}{h} = 4a + 2h.$

□

Example 12. Find and simplify the difference quotient $\frac{f(a+h)-f(a)}{h}$ for the function $f(x) = \frac{1}{2x-1}$.

Solution: We have

$$\frac{f(a+h) - f(a)}{h} = \frac{\frac{1}{2(a+h)-1} - \frac{1}{2a-1}}{h}.$$

We add fractions in the numerator and simplify.

$$\frac{\frac{1}{2(a+h)-1} - \frac{1}{2a-1}}{h} = \frac{2a-1 - (2(a+h)-1)}{(2a-1)(2(a+h)-1)} \cdot \frac{1}{h} = \frac{-2h}{h(2a-1)(2a-1+2h)}.$$

We can reduce the fraction and rearrange the denominator. Then

$$\frac{f(a+h) - f(a)}{h} = \frac{-2}{(2a-1)(2a-1) + 2h(2a-1)} = -\frac{2}{(2a-1)^2 + 2h(2a-1)}.$$

□

Example 13. For given functions $f(x) = x^2 + x$ and $g(x) = -5x$ find $f \pm g$, $f \cdot g$, f/g , $f \circ g$, $g \circ f$, $f \circ f$, $g \circ g$.

Solution:

- $(f + g)(x) = f(x) + g(x) = x^2 + x - 5x = x^2 - 4x,$
- $(f - g)(x) = f(x) - g(x) = x^2 + x - (-5x) = x^2 + 6x,$
- $(f \cdot g)(x) = f(x) \cdot g(x) = (x^2 + x) \cdot (-5x) = -5x^3 - 5x^2,$
- $(f/g)(x) = \frac{f(x)}{g(x)} = \frac{x^2 + x}{-5x} = -\frac{1}{5}(x + 1), x \neq 0,$
- $(f \circ g)(x) = f(g(x)) = (-5x)^2 + (-5x) = 25x^2 - 5x,$
- $(g \circ f)(x) = g(f(x)) = -5 \cdot (x^2 + x) = -5x^2 - 5x,$
- $(f \circ f)(x) = f(f(x)) = (x^2 + x)^2 + x^2 + x = x^4 + 2x^3 + x^2 + x^2 + x = x^4 + 2x^3 + 2x^2 + x,$
- $(g \circ g)(x) = g(g(x)) = -5 \cdot (-5x) = 25x.$





□

Example 14. The cost C (in Czech crowns) to talk m minutes a month on a mobile phone plan is modeled by

$$C(m) = \begin{cases} 399 & \text{if } 0 \leq m \leq 100, \\ 399 + 3.5(m - 100) & \text{if } m > 100. \end{cases}$$

- a) How much does it cost to talk 65 minutes per month with this plan?
 b) How much does it cost to talk 4 hours a month with this plan?

Solution:

- a) $C(65) = 399$, so it costs 399 CZK to talk 65 minutes per month with this plan.
 b) Since 4 hours means 240 minutes, we substitute $m = 240$ and get $C(240) = 889$. It costs 889 Czech crowns to talk 4 hours per month with this plan.

□

1.4 Basic Properties of Functions

Example 15. Determine if the function $f(x) = 3x^4 + 2x^2$ is even, odd, or neither.

Solution: A function f is called **even** if $f(-x) = f(x)$ (**odd** if $f(-x) = -f(x)$) for every x in the domain of f . We will replace x with $-x$ and check these conditions.

$$f(-x) = 3(-x)^4 + 2(-x)^2 = 3x^4 + 2x^2 = f(x).$$

Because $f(-x) = f(x)$, the function f is an even function.

□

Example 16. Determine if the function $f(x) = \frac{4x}{1-x^2}$ is even, odd, or neither.

Solution: We will replace x with $-x$ and check relevant conditions.

$$f(-x) = \frac{4(-x)}{1-(-x)^2} = \frac{-4x}{1-x^2} = -\frac{4x}{1-x^2} = -f(x).$$

Because $f(-x) = -f(x)$, the function f is an odd function.

□

Example 17. Determine if the function $f(x) = \frac{1-4x}{3}$ is one-to-one.

Solution: A function f is said to be **one-to-one** if and only if whenever $f(x_1) = f(x_2)$, then $x_1 = x_2$. We will assume that $f(x_1) = f(x_2)$ and try to deduce $x_1 = x_2$.

$$\begin{aligned} f(x_1) &= f(x_2) \\ \frac{1-4x_1}{3} &= \frac{1-4x_2}{3} \\ 1-4x_1 &= 1-4x_2 \\ x_1 &= x_2 \end{aligned}$$





The function f is one-to-one.

□

Example 18. Find the inverse function of the function $f(x) = \frac{3x}{1+x}$.

Solution: The domain of f is $D_f = \mathbb{R} - \{-1\}$. We determine if f is one-to-one function.

$$\begin{aligned}\frac{3x_1}{1+x_1} &= \frac{3x_2}{1+x_2} \\ 3x_1(1+x_2) &= 3x_2(1+x_1) \\ 3x_1 + 3x_1x_2 &= 3x_2 + 3x_1x_2 \\ 3x_1 &= 3x_2 \\ x_1 &= x_2\end{aligned}$$

The function f is one-to-one, so it is invertible and it has a sense to find the function f^{-1} . We solve the equation $y = f(x)$ for x .

$$\begin{aligned}y &= \frac{3x}{1+x} \\ y(1+x) &= 3x \\ y + yx &= 3x \\ y &= 3x - yx \\ y &= (3 - y)x \\ \frac{y}{3 - y} &= x \\ y &= \frac{x}{3 - x}\end{aligned}$$

We have $y = f^{-1}(x) = \frac{x}{3-x}$. We can verify our result by checking that $(f \circ f^{-1})(x) = x$ for all x in the range of f which is the set $\mathbb{R} - \{3\}$:

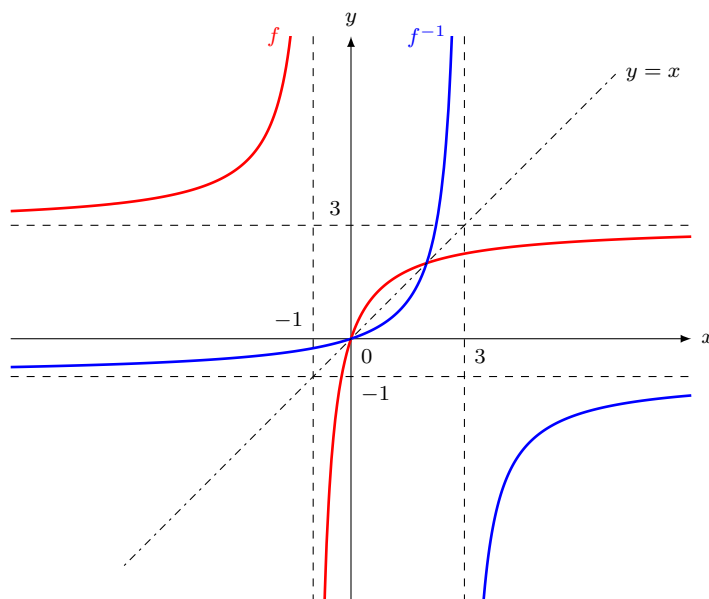
$$(f^{-1} \circ f)(x) = \frac{\frac{3x}{1+x}}{3 - \frac{3x}{1+x}} = \frac{\frac{3x}{1+x}}{\frac{3-3x-3x}{1+x}} = \frac{\frac{3x}{1+x}}{\frac{3}{1+x}} = \frac{3x}{1+x} \cdot \frac{1+x}{3} = x.$$

Further, $(f^{-1} \circ f)(x) = x$ must be true for all $x \in D_f$.

$$(f \circ f^{-1})(x) = \frac{3 \cdot \frac{x}{3-x}}{1 + \frac{x}{3-x}} = \frac{\frac{3x}{3-x}}{\frac{3-x+x}{3-x}} = \frac{\frac{3x}{3-x}}{\frac{3}{3-x}} = \frac{3x}{3-x} \cdot \frac{3-x}{3} = x.$$

We can also verify our result graphically. Here is the graph of the function f and its inverse function. Their graphs are symmetric about the line $y = x$.





□

1.5 Elementary Functions

Example 19. Find the slope of a line if $(7, 4)$ and $(-1, 2)$ are the points on the line.

Solution: Any two points in the Cartesian plane determine a unique line. A line joining two points (x_1, y_1) and (x_2, y_2) has its slope $k = \frac{y_2 - y_1}{x_2 - x_1}$. We divide the change in y by the change in x .

$$k = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 4}{-1 - 7} = \frac{-2}{-8} = \frac{1}{4}.$$

□

Example 20. Write the slope-intercept form of an equation of a line passing through the points $(2, 4)$ and $(4, 7)$.

Solution: The slope-intercept form of an equation of a line takes the form $y = kx + q$ where k is the slope and $(0; q)$ is y -intercept. We use the points $(2, 4)$ and $(4, 7)$ to calculate the slope.

$$k = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 4}{4 - 2} = \frac{3}{2}.$$

We substitute the slope and the coordinates of one of the points into the slope-intercept form.

$$\begin{aligned} y &= kx + q \\ 4 &= \frac{3}{2} \cdot 2 + q \\ 4 &= 3 + q \\ q &= 1. \end{aligned}$$





The slope-intercept form of the line is $y = \frac{3}{2}x + 1$.

□

Example 21. Rewrite the quadratic function $f(x) = 2x^2 - 8x + 3$ in the standard form.

Solution: The standard form of a quadratic function is $f(x) = a(x - x_0)^2 + y_0$, where a , x_0 , y_0 are real numbers and the point (x_0, y_0) is the vertex of the graph $y = f(x)$. To convert the quadratic function $f(x) = 2x^2 - 8x + 3$ to standard form, we complete the square.

$$\begin{aligned}
 f(x) &= 2x^2 - 8x + 3 \\
 &= 2 \left(x^2 - 4x + \frac{3}{2} \right) && \text{Factor out the coefficient of } x^2. \\
 &= 2 \left(x^2 - 4x + (2)^2 - (2)^2 + \frac{3}{2} \right) && \text{Take half of the coefficient of } x, \text{ add and subtract its square.} \\
 &= 2(x^2 - 4x + 4) + 2 \left(-4 + \frac{3}{2} \right) && \text{Group the perfect square trinomial.} \\
 &= 2(x - 2)^2 - 5.
 \end{aligned}$$

□

For the quadratic function $f(x) = ax^2 + bx + c$, where a , b , c are real numbers with $a \neq 0$, the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ is the **vertex of the graph** of $y = f(x)$.

Example 22. Find the vertex of the quadratic function $f(x) = 3x^2 + 5x - 7$.

Solution: The x -coordinate of the vertex will be

$$-\frac{b}{2a} = -\frac{5}{2 \cdot 3} = -\frac{5}{6}.$$

The y -coordinate of the vertex will be

$$f\left(-\frac{b}{2a}\right) = 3\left(-\frac{5}{6}\right)^2 + 5\left(-\frac{5}{6}\right) - 7 = \frac{75}{36} - \frac{25}{6} - 7 = -\frac{109}{12}.$$

The vertex is $\left(-\frac{5}{6}, -\frac{109}{12}\right)$.

□

Example 23. Find the x - and y -intercepts of the quadratic $f(x) = 2x^2 + 5x - 3$.

Solution: To find y -intercept we evaluate $f(0)$

$$f(0) = 2 \cdot 0^2 + 5 \cdot 0 - 3 = -3.$$

The y -intercept is at $(0, -3)$. To find x -intercept we solve the equation $f(x) = 0$:

$$2x^2 + 5x - 3 = 0.$$





One of the methods for solving quadratic equations is to use the quadratic formula. In our case $a = 2$, $b = 5$, $c = -3$, so we have

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 2 \cdot (-3)}}{2 \cdot 2} = \frac{-5 \pm 7}{4}$$

$$x_1 = -3, \quad x_2 = \frac{1}{2}$$

There are the two solutions for given equation, it means there are the two x -intercepts, $(-3, 0)$ and $(\frac{1}{2}, 0)$. □

Example 24. Find the x -intercepts of $f(x) = 2x^6 - 14x^4 + 24x^2$.

Solution: We solve the equation $f(x) = 0$. In our case we can use factoring.

$$\begin{aligned} 2x^6 - 14x^4 + 24x^2 &= 0 && \text{Factor out the greatest common factor.} \\ 2x^2(x^4 - 7x^2 + 12) &= 0 && \text{Factor the trinomial.} \\ 2x^2(x^2 - 3)(x^2 - 4) &= 0 && \text{Set each factor equal to zero.} \\ x^2 &= 0 \quad \text{or} \quad x^2 - 4 = 0 \quad \text{or} \quad x^2 - 3 = 0 \\ x^2 &= 0 \quad \text{or} \quad x^2 = 4 \quad \text{or} \quad x^2 = 3 \\ x &= 0 \quad \text{or} \quad x = \pm 2 \quad \text{or} \quad x = \pm \sqrt{3} \end{aligned}$$

We get five x -intercepts: $(0, 0)$, $(-2, 0)$, $(2, 0)$, $(-\sqrt{3}, 0)$, $(\sqrt{3}, 0)$. □

A **rational function** is a function which is the ratio of polynomial functions. If the degree of the denominator is less than the degree of the numerator, we can perform polynomial long division.

Example 25. Use long division to divide the polynomial $6x^3 + 7x^2 - 20x + 5$ by the polynomial $2x + 3$.

Solution: First we divide the leading term $6x^3$ of the dividend by the leading term $2x$ of the divisor.

$\begin{array}{r} (6x^3 + 7x^2 - 20x + 5) : (2x + 3) = 3x^2 - x - \frac{17}{2} \\ - (6x^3 + 9x^2) \\ \hline -2x^2 - 20x + 5 \\ - (-2x^2 - 3x) \\ \hline -17x + 5 \\ - (-17x - \frac{51}{2}) \\ \hline \frac{61}{2} \end{array}$	<p>$6x^3$ divided by $2x$ is $3x^2$. Multiply $2x + 3$ by $3x^2$ and subtract. $-2x^2$ divided by $2x$ is $-x$. Multiply $2x + 3$ by $-x$ and subtract. $-17x$ divided by $2x$ is $-\frac{17}{2}$. Multiply $2x + 3$ by $-\frac{17}{2}$ and subtract. The remainder is $\frac{61}{2}$.</p>
---	--

We can write the result as

$$\frac{6x^3 + 7x^2 - 20x + 5}{2x + 3} = 3x^2 - x - \frac{17}{2} + \frac{\frac{61}{2}}{2x + 3}$$





or

$$6x^3 + 7x^2 - 20x + 5 = \left(3x^2 - x - \frac{17}{2}\right)(2x + 3) + \frac{61}{2}.$$

□

Example 26. Use the properties of logarithms to simplify the expression $(\ln 24 + \ln(1/3)) / (\ln 16)$.

Solution:

$$\begin{aligned} (\ln 24 + \ln(1/3)) / (\ln 16) &= \frac{\ln(8 \cdot 3) + \ln(3^{-1})}{\ln 16} = \frac{\ln 8 + \ln 3 - \ln 3}{\ln 16} = \frac{\ln 8}{\ln 16} \\ &= \frac{\ln 2^3}{\ln 2^4} = \frac{3 \ln 2}{4 \ln 2} = \frac{3}{4}. \end{aligned}$$

□

Example 27. Evaluate $y = \log_2 32$ without using a calculator.

Solution: We will convert the logarithm into exponential form. The equation $y = \log_2 32$ is equivalent to $2^y = 32$. The base 2 must be raised to the exponent 5 in order to get 32, so $y = 5$.

□

Example 28. Solve the equation $9^{2x+5} = \frac{1}{81^{x-2}}$.

Solution: We will simplify given equation to the form where both sides are as powers with a common base. Then we will use the one-to-one property of exponential functions to solve the equation for x .

$9^{2x+5} = \frac{1}{81^{x-2}}$	
$(3^2)^{2x+5} = (3^{-4})^{x-2}$	Rewrite both sides as powers with base 3.
$3^{4x+10} = 3^{-4x+8}$	Use the one-to-one property of exponential functions.
$4x + 10 = -4x + 8$	Add $4x$ and -10 to both sides.
$8x = -2$	Divide by 8.
$x = -\frac{2}{8} = -\frac{1}{4}$	

The solution is $x = -\frac{1}{4}$.

□

Example 29. Solve the equation $4 \cdot 7^{-3x} = 20$.

Solution: We will use the properties of exponentials and logarithms to solve this equation.

$4 \cdot 7^{-3x} = 20$	
$7^{-3x} = 5$	Divide both sides by 4.
$\ln(7^{-3x}) = \ln 5$	Use the natural logarithm.
$-3x \ln 7 = \ln 5$	Use the Power Rule for logarithms.
$x = -\frac{\ln 5}{3 \ln 7}$	Divide both sides by $-3 \ln 7$.





The solution is $x = -\frac{\ln 5}{3 \ln 7}$.

□

Example 30. Convert the angle 240° from degrees to radians.

Solution: The relationship between degrees and radians is $1 \text{ degree} = \frac{\pi}{180}$ ($\doteq 0.017$) radians. Hence

$$240^\circ = 240 \cdot \frac{\pi}{180} = \frac{4}{3}\pi.$$

□

Example 31. If $\sin \alpha = \frac{3}{4}$ and α is in the second quadrant, find $\cos \alpha$.

Solution: We can use the formula known as the Pythagorean Identity

$$\cos^2 \alpha + \sin^2 \alpha = 1.$$

Substituting $\sin \alpha = \frac{3}{4}$ into the Pythagorean Identity we get

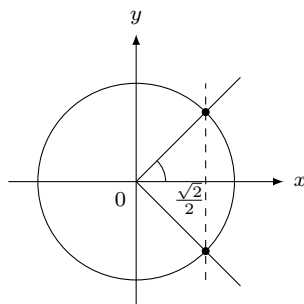
$$\begin{aligned} \cos^2 \alpha + \sin^2 \alpha &= 1 \\ \cos^2 \alpha + \frac{9}{16} &= 1 \\ \cos^2 \alpha &= \frac{7}{16} \\ \cos \alpha &= \pm \sqrt{\frac{7}{16}} = \pm \frac{\sqrt{7}}{4}. \end{aligned}$$

Since α is in the second quadrant and x -coordinates are negative in the second quadrant, $\cos \alpha$ is negative too. So $\cos \alpha = -\frac{\sqrt{7}}{4}$.

□

Example 32. Find all of the angles which satisfy the equation $\cos \alpha = \frac{\sqrt{2}}{2}$.

Solution: Let us consider the Unit Circle. The terminal side of α intersects this circle for $x = \frac{\sqrt{2}}{2}$.



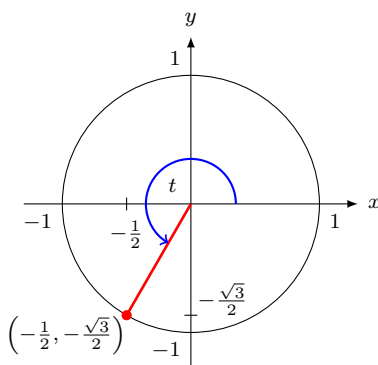


One solution $\alpha = \frac{\pi}{4}$ is in the first quadrant and the second solution $\alpha = -\frac{\pi}{4}$ in the fourth quadrant. All other solutions must be coterminal with $-\frac{\pi}{4}$ or $\frac{\pi}{4}$ due to period 2π of the cosine function, so all the solutions are of the form

$$-\frac{\pi}{4} + 2k\pi \quad \text{or} \quad \frac{\pi}{4} + 2k\pi \quad \text{where} \quad k \in \mathbb{Z}.$$

□

Example 33. Find the values of the four trigonometric functions of the angle t based on the following figure.



Solution:

$$\sin t = y = -\frac{\sqrt{3}}{2}$$

$$\cos t = x = -\frac{1}{2}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$$

$$\cot t = \frac{\cos t}{\sin t} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

□

Let f be one of the functions

$$f(x) = a \cdot \sin(b \cdot x + c) \quad \text{or} \quad f(x) = a \cdot \cos(b \cdot x + c).$$

The number $|a|$ is called the **amplitude**, the number $|\frac{2\pi}{b}|$ is the **period** and the number $\frac{-c}{b}$ is called the **phase shift**.

Example 34. Find the amplitude, period, and phase-shift for the function $f(x) = -2 \cdot \sin(\frac{x}{2} + 5)$.

Solution: The amplitude is $|a| = |-2| = 2$. The period is $|\frac{2\pi}{b}| = |\frac{2\pi}{\frac{1}{2}}| = 4\pi$. The phase-shift is $\frac{-c}{b} = \frac{-5}{\frac{1}{2}} = -10$.

□

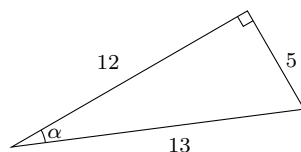


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Example 35. Using the triangle shown in the following figure, evaluate $\sin \alpha$, $\cos \alpha$, $\tan \alpha$ and $\cot \alpha$.



Solution:

$$\sin \alpha = \frac{\text{opposite } \alpha}{\text{hypotenuse}} = \frac{5}{13}$$

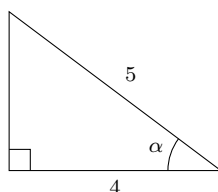
$$\cos \alpha = \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{12}{13}$$

$$\tan \alpha = \frac{\text{opposite } \alpha}{\text{adjacent to } \alpha} = \frac{5}{12}$$

$$\cot \alpha = \frac{\text{adjacent to } \alpha}{\text{opposite } \alpha} = \frac{12}{5}.$$

□

Example 36. Solve the triangle in the following figure for the angle α .



Solution:

$$\cos \alpha = \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{4}{5}$$

$$\alpha = \arccos \frac{4}{5}$$

$$\alpha = 0.6435 \text{ or about } 36.87^\circ.$$

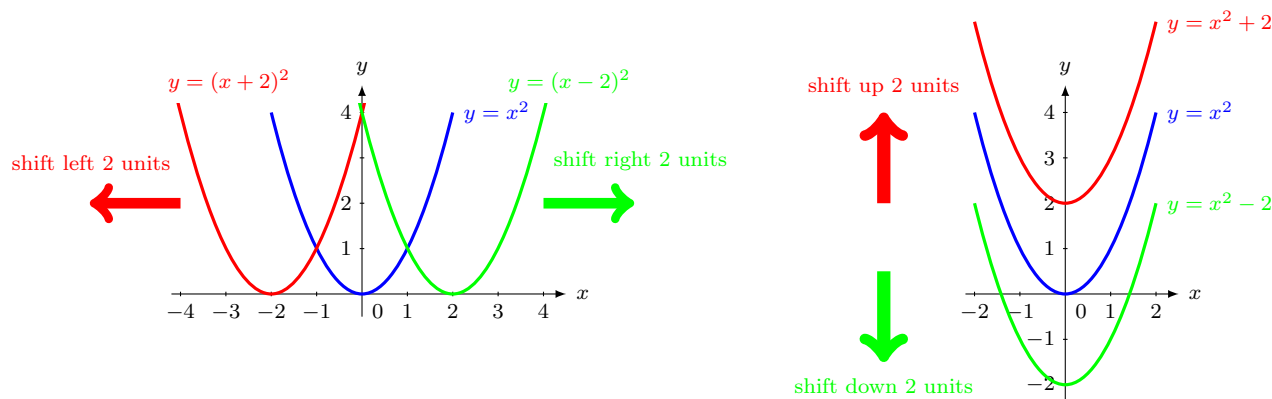
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1.6 Transformations

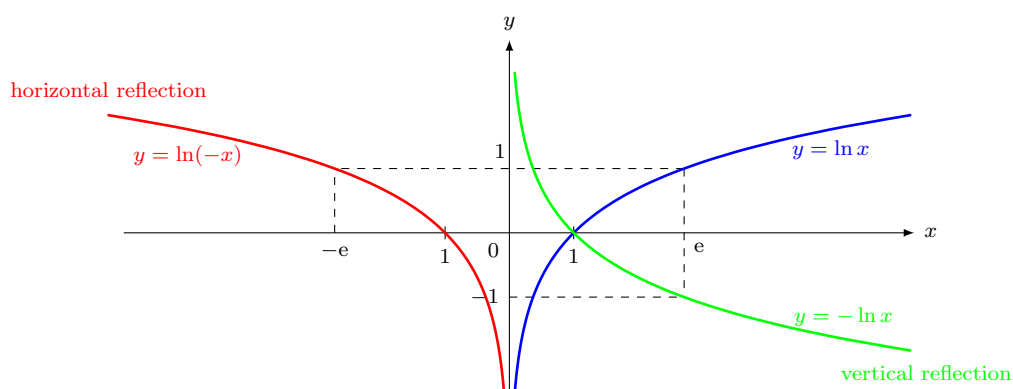
We will present several kinds of the graph modifications involving shifting the entire graph of a function up, down, right, or left. Further we will apply a reflection over the x - or y -axis to a function and also a horizontal and a vertical stretch and compression.

The graph of $y = f(x + c)$ is the graph of the function f **shifted left** c units. The graph of $y = f(x - c)$ is the graph of the function f **shifted right** c units. The graph of $y = f(x) + c$ is the graph of the function f **shifted up** c units. The graph of $y = f(x) - c$ is the graph of the function f **shifted down** c units.

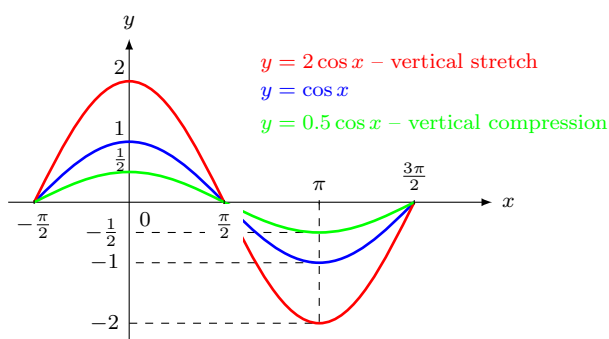




The graph of $y = -f(x)$ is the graph of the function f **reflected across the x -axis** (a vertical reflection). The graph of $y = f(-x)$ is the graph of the function f **reflected across the y -axis** (a horizontal reflection).

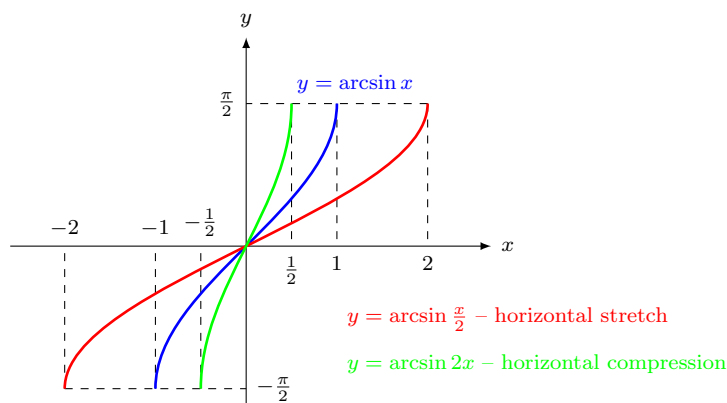


The graph of $y = cf(x)$ for $c > 1$ is the graph of the function f **stretched (expanded) vertically** by a factor of c . The graph of $y = cf(x)$ for $0 < c < 1$ is the graph of the function f **compressed (shrunk) vertically** by a factor of $1/c$.



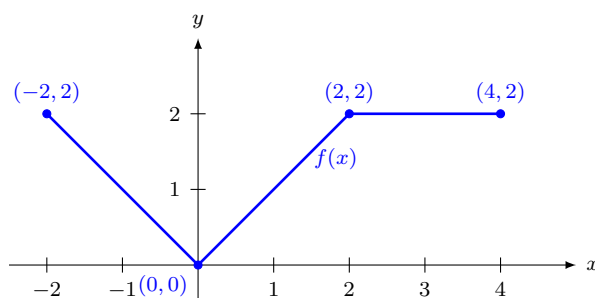


The graph of $y = f(cx)$ for $c > 1$ is the graph of the function f **compressed (shrunk) horizontally** by a factor of c . The graph of $y = f(cx)$ for $0 < c < 1$ is the graph of the function f **stretched (expanded) horizontally** by a factor of $1/c$.

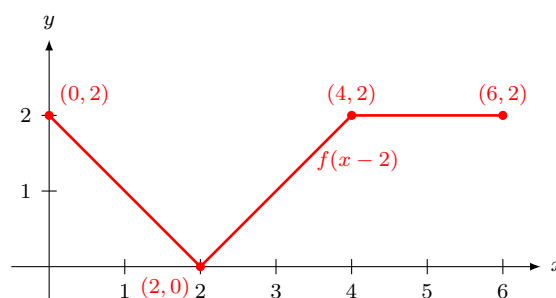
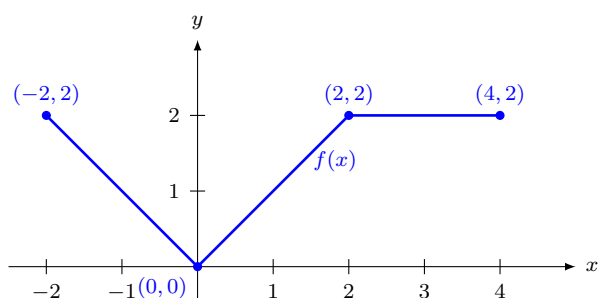


Example 37. Use the graph of $f(x)$ to sketch the graphs of the following functions:

a) $f(x-2)$, b) $f(x+1)$, c) $f(x)-2$, d) $f(x)+1$, e) $f(-x)$, f) $-f(x)$, g) $2f(x)$, h) $\frac{1}{2}f(x)$, if $f(x)$ is given by the figure:

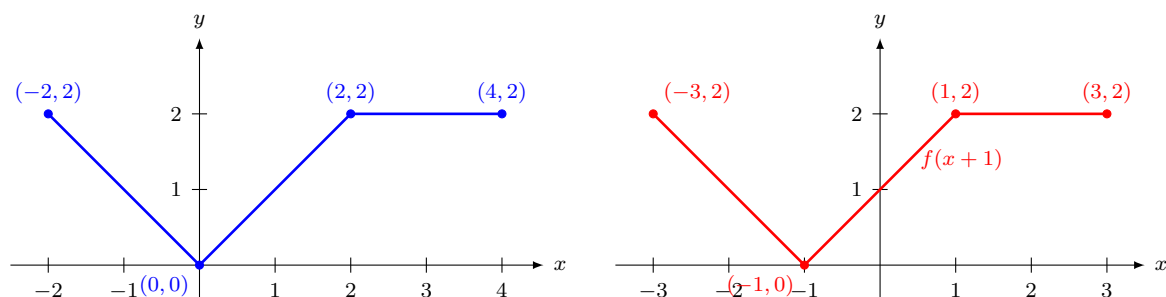


Solution: To sketch the graph of $f(x-2)$ we shift the graph of f to the right by 2. We add 2 to the x -coordinates of the points on the graph of f . The result is below.

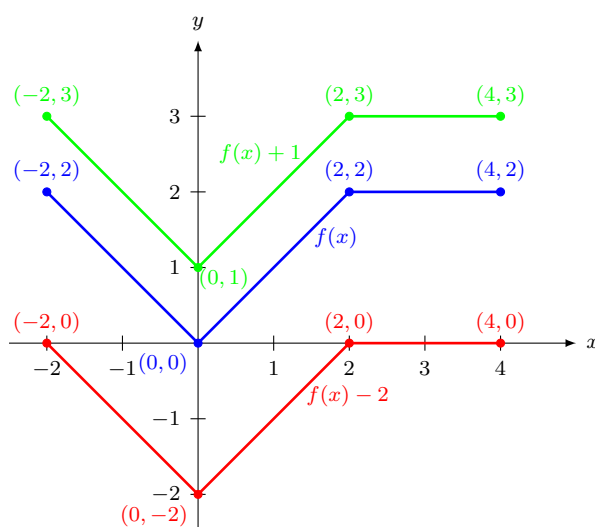


To sketch the graph of $f(x+1)$ we shift the graph of f to the left by 1. We subtract 1 from the x -coordinates of the points on the graph of f . The result is below.

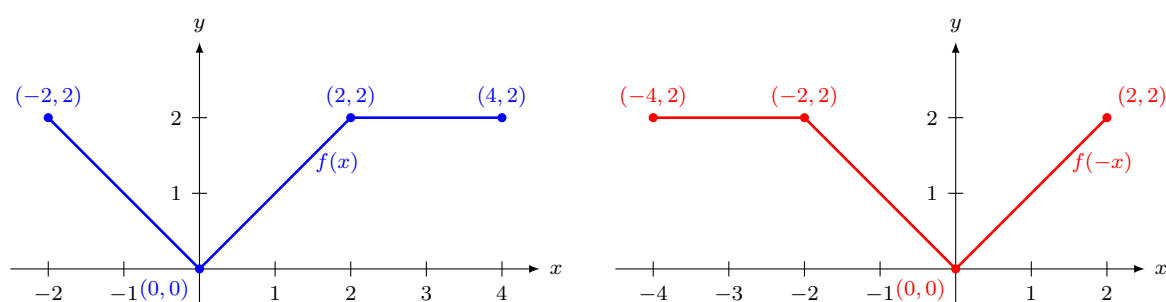




To sketch the graph of $f(x) - 2$ we shift the graph of f vertically down 2 units. We add -2 to the y -coordinates of the points on the graph of f . To sketch the graph of $f(x) + 1$ we shift the graph of f vertically up 1 unit. We add 1 to the y -coordinates of the points on the graph of f . The results are below.

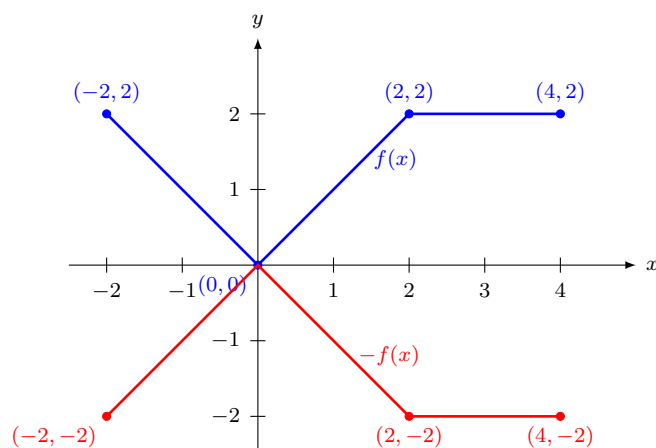


To obtain the graph of $y = f(-x)$ we reflect the graph of f across the y -axis. It is so called a **horizontal reflection** of the function f .

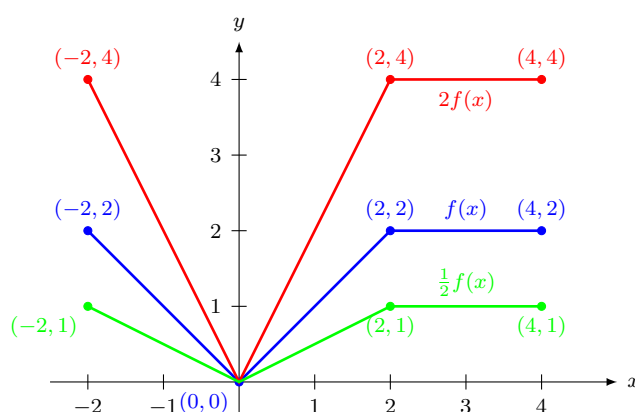


The graph of $y = -f(x)$ can be sketch by reflecting the graph of f about the x -axis. It is so called a **vertical reflection** of the function f .





To obtain the graph of $y = 2f(x)$ we multiply all of the y -coordinates of the points on the graph of f by 2. To obtain the graph of $y = \frac{1}{2}f(x)$ we multiply all of the y -coordinates of the points on the graph of f by $\frac{1}{2}$. It is so called a **vertical scaling** of the function f .



□

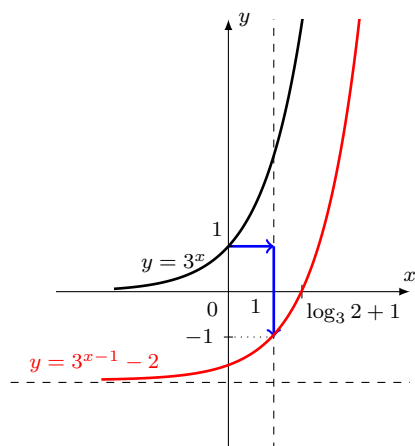
Example 38. Use the graph of $f(x) = 3^x$ to sketch the graph of the function $g(x) = f(x - 1) - 2$.

Solution: The domain of g is $(-\infty, \infty)$. The graph of f will be transformed in two ways. The formula $f(x - 1)$ corresponds to a shift to the left by 1 and the subtraction by 2 in $f(x - 1) - 2$ means a vertical shift down by 2. It is useful to find the points of intersection of the graph of g and the axes.

$$\begin{aligned} x = 0 &\Rightarrow y = 3^{-1} - 2 \Rightarrow y = -\frac{5}{3} \Rightarrow \left[0, -\frac{5}{3}\right] \\ y = 0 &\Rightarrow 3^{x-1} - 2 = 0 \Rightarrow 3^{x-1} = 2 \Rightarrow x - 1 = \log_3 2 \Rightarrow \\ &\Rightarrow x = \log_3 2 + 1 \Rightarrow [\log_3 2 + 1, 0] \end{aligned}$$

The graph of the function $g(x) = f(x - 1) - 2 = 3^{x-1} - 2$ is illustrated in the following figure.





□



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2 Limits and Continuity

We will evaluate the limit of a function at a given point.

2.1 Evaluating Limits

Example 39. Evaluate $\lim_{x \rightarrow 2} \frac{x-4}{7x-5}$.

Solution: The function $f(x) = \frac{x-4}{7x-5}$ is continuous at the point 2. We get the limit by evaluating the function f at the point 2 while using the properties of limits.

$$\lim_{x \rightarrow 2} \frac{x-4}{7x-5} = \frac{2-4}{7 \cdot 2 - 5} = \frac{-2}{9} = -\frac{2}{9}.$$

□

Example 40. Evaluate $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin 2x - \cos x + 2}{\sin^2 x}$.

Solution: Because of continuity we only evaluate given function at $\frac{\pi}{4}$ by using the properties of limits.

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin 2x - \cos x + 2}{\sin^2 x} = \frac{\sin 2\frac{\pi}{4} - \cos \frac{\pi}{4} + 2}{\sin^2 \frac{\pi}{4}} = \frac{1 - \frac{\sqrt{2}}{2} + 2}{\left(\frac{\sqrt{2}}{2}\right)^2} = \frac{3 - \frac{\sqrt{2}}{2}}{\frac{1}{2}} = 6 - \sqrt{2}.$$

□

Example 41. Evaluate $\lim_{x \rightarrow 2} \frac{x^2 + 3x - 10}{x - 2}$.

Solution: If we plug $x = 2$ into the given function, we get

$$\lim_{x \rightarrow 2} \frac{x^2 + 3x - 10}{x - 2} = \frac{0}{0}.$$

We need to simplify the function as much as possible. Both the numerator and the denominator of a rational function are zero at $x = 2$, so the factor $(x - 2)$ can be factored and after that cancelled from both the numerator and the denominator. The limit is

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^2 + 3x - 10}{x - 2} &= \lim_{x \rightarrow 2} \frac{(x-2)(x+5)}{x-2} && \text{Factor the numerator.} \\ &= \lim_{x \rightarrow 2} \frac{\cancel{(x-2)}(x+5)}{\cancel{x-2}} && \text{Cancel the common factors.} \\ &= \lim_{x \rightarrow 2} (x+5) && \text{Evaluate.} \\ &= 2 + 5 = 7. \end{aligned}$$





□

Example 42. Evaluate $\lim_{x \rightarrow 3} \frac{x^2 + 4x - 21}{3 - x}$.

Solution: The numerator and denominator are zeros at the point 3. We need to divide any factors common to the numerator and denominator.

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{x^2 + 4x - 21}{3 - x} &= \left[\frac{0}{0} \right] = \lim_{x \rightarrow 3} \frac{(x - 3)(x + 7)}{-(x - 3)} && \text{Factor the numerator.} \\ &= \lim_{x \rightarrow 3} \frac{\cancel{(x - 3)}(x + 7)}{-\cancel{(x - 3)}} && \text{Cancel the common factors.} \\ &= \lim_{x \rightarrow 3} \frac{x + 7}{-1} = -\lim_{x \rightarrow 3} (x + 7) && \text{Evaluate.} \\ &= -(3 + 7) = -10. \end{aligned}$$

□

Example 43. Evaluate $\lim_{x \rightarrow 0} \frac{x}{\sqrt{49 - x} - 7}$.

Solution: Given limit of a function is in indeterminate $\left[\frac{0}{0} \right]$ form at 0 and contains a root. We will multiply both numerator and denominator by the conjugate of the expression involving the square root. The conjugate of $\sqrt{49 - x} - 7$ is $\sqrt{49 - x} + 7$.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x}{\sqrt{49 - x} - 7} &= \lim_{x \rightarrow 0} \frac{x}{\sqrt{49 - x} - 7} \cdot \frac{\sqrt{49 - x} + 7}{\sqrt{49 - x} + 7} && \text{Multiply numerator and denominator} \\ &&& \text{by the conjugate.} \\ &= \lim_{x \rightarrow 0} \frac{x \cdot (\sqrt{49 - x} + 7)}{(49 - x) - 49} && \text{Do not multiply out the numerator.} \\ &= \lim_{x \rightarrow 0} \frac{x \cdot (\sqrt{49 - x} + 7)}{-x} && \text{Cancel.} \\ &= \lim_{x \rightarrow 0} \frac{\sqrt{49 - x} + 7}{-1} && \text{Evaluate.} \\ &= \frac{\sqrt{49 - 0} + 7}{-1} = -(7 + 7) = -14. \end{aligned}$$

□

Example 44. Evaluate $\lim_{x \rightarrow 3} \frac{\sqrt{5x + 1} - 4}{x - 3}$.

Solution: The limit is the type 0/0. If the numerator includes a root we rationalize the numerator by multiplying both the numerator and denominator by the conjugate of the numerator (in a similar way for the denominator).





$$\begin{aligned}
 \lim_{x \rightarrow 3} \frac{\sqrt{5x+1}-4}{x-3} &= \lim_{x \rightarrow 3} \frac{\sqrt{5x+1}-4}{x-3} \cdot \frac{\sqrt{5x+1}+4}{\sqrt{5x+1}+4} && \text{Multiply numerator and denominator by the conjugate.} \\
 &= \lim_{x \rightarrow 3} \frac{(5x+1)-16}{(x-3) \cdot (\sqrt{5x+1}+4)} && \text{Do not multiply out the numerator.} \\
 &= \lim_{x \rightarrow 3} \frac{5(x-3)}{(x-3) \cdot (\sqrt{5x+1}+4)} && \text{Cancel.} \\
 &= \lim_{x \rightarrow 3} \frac{5}{\sqrt{5x+1}+4} && \text{Evaluate.} \\
 &= \frac{5}{\sqrt{15+1}+4} = \frac{5}{8}.
 \end{aligned}$$

□

Example 45. Evaluate $\lim_{x \rightarrow \infty} (2x^5 - 7x^2 + 3)$.

Solution: If we plug minus infinity into the polynomial we get the following

$$\lim_{x \rightarrow \infty} (2x^5 - 7x^2 + 3) = \infty - \infty + 3.$$

This is one of indeterminate forms. We can try to factor the largest power of x out of the polynomial.

$$\begin{aligned}
 \lim_{x \rightarrow \infty} (2x^5 - 7x^2 + 3) &= \lim_{x \rightarrow \infty} x^5 \left(2 - \frac{7}{x^3} + \frac{3}{x^5} \right) && \text{Factor the largest power of } x \text{ out.} \\
 &= \lim_{x \rightarrow \infty} x^5 \cdot \lim_{x \rightarrow \infty} \left(2 - \frac{7}{x^3} + \frac{3}{x^5} \right) && \text{Evaluate a limit of each of the terms.} \\
 &= \infty \cdot (2 - 0 + 0) && \text{Evaluate.} \\
 &= \infty.
 \end{aligned}$$

□

When evaluating limits at infinities for polynomials we can use the following rule: If $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_x + a_0$ is a polynomial of a degree n (i.e. $a_n \neq 0$) then,

$$\lim_{x \rightarrow \pm\infty} p(x) = \lim_{x \rightarrow \pm\infty} a_n x^n.$$

Example 46. Evaluate $\lim_{x \rightarrow -\infty} (7x^3 + 5x^2 + 3x - 1)$.

Solution: Using previous fact the limit is

$$\lim_{x \rightarrow -\infty} (7x^3 + 5x^2 + 3x - 1) = \lim_{x \rightarrow -\infty} 7x^3 = 7 \cdot (-\infty) = -\infty.$$

□





Example 47. Evaluate $\lim_{x \rightarrow \infty} \frac{4x^4 - 2x^2 + 2x}{-2x^4 + 3x - 1}$.

Solution: Using previous rule for polynomials we have

$$\lim_{x \rightarrow \infty} \frac{4x^4 - 2x^2 + 2x}{-2x^4 + 3x - 1} = \lim_{x \rightarrow \infty} \frac{\infty}{-\infty},$$

which is indeterminate form. Since both the numerator and denominator are polynomials, we can factor the largest power of x that is in the denominator from both the denominator and the numerator.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{4x^4 - 2x^2 + 2x}{-2x^4 + 3x - 1} &= \lim_{x \rightarrow \infty} \frac{x^4 \left(4 - \frac{2}{x^2} + \frac{2}{x^3}\right)}{x^4 \left(-2 + \frac{3}{x^3} - \frac{1}{x^4}\right)} \\ &= \lim_{x \rightarrow \infty} \frac{4 - \frac{2}{x^2} + \frac{2}{x^3}}{-2 + \frac{3}{x^3} - \frac{1}{x^4}} \\ &= \frac{4 - 0 + 0}{-2 + 0 - 0} \\ &= -2. \end{aligned}$$

Factor the largest power of x in the denominator out from both the denominator and the numerator.

Cancel x^4 and find the limit of the remaining terms.

Evaluate.

□

For a rational function of the following form

$$R(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0},$$

where the $a_n \neq 0$, $b_m \neq 0$, it can be derived the following rule

1. If $n = m$, then $\lim_{x \rightarrow -\infty} R(x) = \lim_{x \rightarrow \infty} R(x) = \frac{a_n}{b_m}$.
2. If $n > m$, then $\lim_{x \rightarrow -\infty} R(x) = \lim_{x \rightarrow \infty} R(x) = +\infty$ or $-\infty$.
3. If $n < m$, then $\lim_{x \rightarrow -\infty} R(x) = \lim_{x \rightarrow \infty} R(x) = 0$.

Example 48. Evaluate $\lim_{x \rightarrow \infty} \frac{7x^3 - 5x^2 - 1}{-2x^2 + 4x + 3}$.

Solution: We can see that the highest power of x is in the numerator. Then the limit will be $+\infty$ or $-\infty$. With respect to the terms with the highest power of x we get

$$\lim_{x \rightarrow \infty} \frac{7x^3 - 5x^2 - 1}{-2x^2 + 4x + 3} = \lim_{x \rightarrow \infty} \frac{7x^3}{-2x^2} = \lim_{x \rightarrow \infty} \frac{7x}{-2} = -\infty.$$

□





Now, we will evaluate limits of the type $L/0$. Let us suppose that $\lim_{x \rightarrow x_0} f(x) = L \in \mathbb{R}$, $L \neq 0$, and $\lim_{x \rightarrow x_0} g(x) = 0$. If there exists reduced neighbourhood $P(x_0)$ of the point x_0 so that $\frac{f(x)}{g(x)} > 0$ for all $x \in P(x_0)$ then

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = +\infty.$$

If there exists reduced neighbourhood $P(x_0)$ of the point x_0 so that $\frac{f(x)}{g(x)} < 0$ for all $x \in P(x_0)$, then

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = -\infty.$$

If the function $\frac{f(x)}{g(x)} > 0$ has different signs from the left side and from the right side of x_0 , then the limit of the type $L/0$ does not exist. If the two one-sided limits have different values then the normal limit will not exist.

Example 49. Evaluate $\lim_{x \rightarrow 4} \frac{2x+5}{x(x-4)}$.

Solution: After the substitution of the point $x = 4$, we see that this limit has the form $13/0$. We need to evaluate one-sided limits. If $x > 4$, then $\frac{2x+5}{x(x-4)} > 0$. The limit at 4 from the right is $+\infty$.

If $x < 4$, then $\frac{2x+5}{x(x-4)} < 0$. The limit at 4 from the left is $-\infty$. Since

$$\lim_{x \rightarrow 4^-} \frac{2x+5}{x(x-4)} \neq \lim_{x \rightarrow 4^+} \frac{2x+5}{x(x-4)},$$

the given limit does not exist. □

Example 50. Evaluate $\lim_{x \rightarrow 3} \frac{4x}{x^2-9}$.

Solution: After substituting the point $x = 3$ into $\frac{4x}{x^2-9}$, we get division by zero, our limit has the form $12/0$.

$$\lim_{x \rightarrow 3} \frac{4x}{x^2-9} = \left[\frac{12}{0} \right] \rightarrow \begin{cases} -3 < x < 3 & \Rightarrow \frac{4x}{x^2-9} < 0 \Rightarrow \lim_{x \rightarrow 3^-} \frac{4x}{x^2-9} = -\infty, \\ x > 3 & \Rightarrow \frac{4x}{x^2-9} > 0 \Rightarrow \lim_{x \rightarrow 3^+} \frac{4x}{x^2-9} = +\infty. \end{cases}$$

The given limit does not exist. □

Example 51. Evaluate $\lim_{x \rightarrow 0} \frac{2}{x^3+5x^2}$.

Solution: As $x \rightarrow 0$ both from the left and from the right, the denominator approaches 0. The limit has the form $2/0$. We can rearrange the given function.

$$\lim_{x \rightarrow 0} \frac{2}{x^3+5x^2} = \lim_{x \rightarrow 0} \frac{2}{x^2(x+5)} = \lim_{x \rightarrow 0} \frac{2}{x+5} \cdot \frac{1}{x^2}.$$





Since

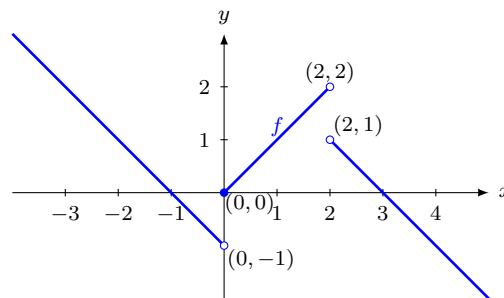
$$\lim_{x \rightarrow 0} \frac{2}{x+5} = \frac{2}{5} \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty \quad (x^2 > 0 \text{ for all } x \neq 0),$$

then

$$\lim_{x \rightarrow 0} \frac{2}{x^3 + 5x^2} = +\infty.$$

□

Example 52. For the function $f(x) = \begin{cases} -x-1 & \text{if } x < 0 \\ x & \text{if } x \in \langle 0, 2 \rangle \\ -x+3 & \text{if } x > 2 \end{cases}$ (see the following figure)



determine each of the following:

- a) $f(0)$ b) $\lim_{x \rightarrow 0^-} f(x)$ c) $\lim_{x \rightarrow 0^+} f(x)$ d) $\lim_{x \rightarrow 0} f(x)$
 e) $f(2)$ f) $\lim_{x \rightarrow 2^-} f(x)$ g) $\lim_{x \rightarrow 2^+} f(x)$ h) $\lim_{x \rightarrow 2} f(x)$
 i) $\lim_{x \rightarrow -\infty} f(x)$ k) $\lim_{x \rightarrow +\infty} f(x)$.

Solution: To determine these values and limits we can use both the definition and the graph of the function f .

- a) Definition says that $f(0) = 0$, the function will take on the y value where the closed dot is.
 b) To see that $\lim_{x \rightarrow 0^-} f(x) = -1$, we can move the test point with x -coordinate approaching 0 from the left along the graph.
 c) $\lim_{x \rightarrow 0^+} f(x) = 0$.
 d) $\lim_{x \rightarrow 0} f(x)$ does not exist, the two one-sided limits are different.
 e) $f(2)$ does not exist.
 f) $\lim_{x \rightarrow 2^-} f(x) = 2$.
 g) $\lim_{x \rightarrow 2^+} f(x) = 1$.
 h) $\lim_{x \rightarrow 2} f(x)$ does not exist.
 i) $\lim_{x \rightarrow -\infty} f(x) = +\infty$.
 j) $\lim_{x \rightarrow +\infty} f(x) = -\infty$.

□



3 Differential Calculus

The concept of Derivative is a major topic of this chapter. We will differentiate the functions of one variable and investigate their properties using derivatives.

3.1 Definition of the Derivative

In this section we will determine the derivative of the function f at x_0 using the definition of the derivative. Let $x_0 \in D_f$. The derivative of the function f at x_0 is defined by

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Alternatively, we also define

$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Example 53. Find the derivative of the function $f(x) = 2x^2 - 3x + 1$ at $x_0 = 3$ using the definition of the derivative.

Solution: We substitute the given function and x_0 into the respective formula.

$f'(3) = \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3}$	Use the definition.
$= \lim_{x \rightarrow 3} \frac{(2x^2 - 3x + 1) - 10}{x - 3}$	Substitute $f(x)$ and $f(3)$.
$= \lim_{x \rightarrow 3} \frac{(2x + 3)(x - 3)}{x - 3}$	Simplify and factor the numerator.
$= \lim_{x \rightarrow 3} (2x + 3)$	Evaluate.
$= 9.$	

□

Example 54. Find the derivative of the function $f(x) = \sqrt{x - 2}$ at $x_0 = 6$ using the definition of the derivative.

Solution: We substitute the given function and x_0 into the respective formula.



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$$\begin{aligned}
 f'(6) &= \lim_{h \rightarrow 0} \frac{f(6+h) - f(6)}{h} && \text{Use the definition.} \\
 &= \lim_{x \rightarrow 0} \frac{\sqrt{4+h} - 2}{h} && \text{Substitute } f(x) \text{ and } f(6). \\
 &= \lim_{x \rightarrow 0} \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} && \text{Multiply the numerator and denominator by the conjugate.} \\
 &= \lim_{h \rightarrow 0} \frac{(4+h) - 4}{h \cdot (\sqrt{4+h} + 2)} && \text{Do not multiply out the numerator.} \\
 &= \lim_{h \rightarrow 0} \frac{h}{h \cdot (\sqrt{4+h} + 2)} && \text{Cancel.} \\
 &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{4+h} + 2} && \text{Evaluate.} \\
 &= \frac{1}{\sqrt{4+0} + 2} = \frac{1}{4}.
 \end{aligned}$$

□

3.2 Differentiation Formulas

In this section, we will only summarize differentiation formulas for basic elementary functions. These formulas can be derived from the definition of the derivative.

$$\begin{aligned}
 (c)' &= 0, & x &\in (-\infty, \infty) \\
 (x^n)' &= nx^{n-1}, \quad n > 0 & x &\in (-\infty, \infty) && \text{(the power rule)} \\
 (x^k)' &= kx^{k-1}, \quad k < 0 & x &\in (-\infty, 0) \cup (0, \infty) \\
 (x^s)' &= sx^{s-1}, \quad s \in \mathbb{R} & x &\in (0, \infty) \\
 (e^x)' &= e^x, & x &\in (-\infty, \infty) \\
 (a^x)' &= a^x \ln a, & x &\in (-\infty, \infty) \\
 (\ln x)' &= \frac{1}{x}, & x &\in (0, \infty) \\
 (\log_a x)' &= \frac{1}{x \ln a}, & x &\in (0, \infty) \\
 (\sin x)' &= \cos x, & x &\in (-\infty, \infty) \\
 (\cos x)' &= -\sin x, & x &\in (-\infty, \infty) \\
 (\operatorname{tg} x)' &= \frac{1}{\cos^2 x}, & x &\in \mathbb{R} - \bigcup_{k \in \mathbb{Z}} \left\{ (2k-1)\frac{\pi}{2} \right\} \\
 (\operatorname{cotg} x)' &= -\frac{1}{\sin^2 x}, & x &\in \mathbb{R} - \bigcup_{k \in \mathbb{Z}} \{k\pi\} \\
 (\arcsin x)' &= \frac{1}{\sqrt{1-x^2}}, & x &\in (-1, 1)
 \end{aligned}$$





$$\begin{aligned}
(\arccos x)' &= -\frac{1}{\sqrt{1-x^2}}, & x \in (-1, 1) \\
(\operatorname{arctg} x)' &= \frac{1}{1+x^2}, & x \in (-\infty, \infty) \\
(\operatorname{arccotg} x)' &= -\frac{1}{1+x^2}, & x \in (-\infty, \infty)
\end{aligned}$$

3.3 Differentiation Rules

We will show the next rules for differentiation. We will use them together with formulas from the previous section to differentiate more complicated functions.

Let f, g be functions and $c \in \mathbb{R}$ be a real constant. The following relations hold

$$\begin{aligned}
(c \cdot f(x))' &= c \cdot f'(x) && \text{the constant multiple rule,} \\
(f(x) \pm g(x))' &= f'(x) \pm g'(x) && \text{the sum rule,} \\
(f(x) \cdot g(x))' &= f'(x) \cdot g(x) + f(x) \cdot g'(x) && \text{the product rule,} \\
\left(\frac{f(x)}{g(x)}\right)' &= \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)} && \text{the quotient rule,}
\end{aligned}$$

whenever the derivatives on the right-hand side exist and the expression on the right-hand side is well defined.

Example 55. Differentiate the function $f(x) = 7 \cdot x^5$.

Solution: Using the constant multiple rule we factor the constant 7 out and then do the derivative.

$$f'(x) = (7 \cdot x^5)' = 7 \cdot (x^5)' = 7 \cdot 5x^4 = 35x^4.$$

□

Example 56. Differentiate the function $f(x) = 2 \cdot 3^x + 4x^8$.

Solution: We compute the derivative of the sum. We need to use the sum rule and the constant multiple rule together.

$$\begin{aligned}
f'(x) &= (2 \cdot 3^x + 4x^8)' \\
&= (2 \cdot 3^x)' + (4x^8)' && \text{Apply the sum rule.} \\
&= 2 \cdot (3^x)' + 4 \cdot (x^8)' && \text{Apply the constant multiple rule.} \\
&= 2 \cdot 3^x \ln 3 + 4 \cdot 8x^7 && \text{Apply differentiation formulas.} \\
&= 2 \ln 3 \cdot 3^x + 32x^7. && \text{Simplify.}
\end{aligned}$$

□

Example 57. Differentiate the function $f(x) = 5e^x - \frac{4}{x^2}$.

Solution: We apply the rule for differentiating the sum of two functions and the constant multiple rule again.

$$\left(5e^x - \frac{4}{x^2}\right)' = 5 \cdot (e^x)' - 4 \cdot (x^{-2})' = 5 \cdot e^x - 4 \cdot (-2)x^{-3} = 5e^x + \frac{8}{x^3}.$$





□

Example 58. Differentiate the function $f(x) = x^6 \cdot \ln x$.

Solution: We use the product rule. We take the derivative of the first function x^6 times the second function $\ln x$ and then add on to that the first function times the derivative of the second function.

$$(x^6 \cdot \ln x)' = (x^6)' \cdot \ln x + x^6 \cdot (\ln x)' = 6x^5 \cdot \ln x + x^6 \cdot \frac{1}{x} = 6x^5 \cdot \ln x + x^5.$$

□

Example 59. Differentiate the function $f(x) = 8x \cdot \cos x$. Evaluate the derivative at $x_0 = \frac{\pi}{2}$.

Solution: We use the product rule to differentiate the function f .

$$(8x \cdot \cos x)' = (8x)' \cdot \cos x + 8x \cdot (\cos x)' = 8 \cos x + 8x \cdot (-\sin x) = 8 \cos x - 8x \sin x.$$

At $x_0 = \frac{\pi}{2}$, we get

$$f' \left(\frac{\pi}{2} \right) = 8 \cos \frac{\pi}{2} - 8 \cdot \frac{\pi}{2} \cdot \sin \frac{\pi}{2} = 8 \cdot 0 - \frac{8\pi}{2} \cdot 1 = -\frac{8\pi}{2}.$$

□

Example 60. Differentiate the function $h(x) = \frac{7x^4}{3x+2}$.

Solution: If we set $f(x) = 7x^4$, $g(x) = 3x + 2$, we can use the quotient rule. It is the derivative of the numerator times the denominator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)} = \frac{7 \cdot 4x^3 \cdot (3x+2) - 7x^4 \cdot 3}{(3x+2)^2} = \frac{84x^4 + 56x^3 - 21x^4}{(3x+2)^2} = \frac{63x^4 + 56x^3}{(3x+2)^2}.$$

□

Example 61. Differentiate the function $f(x) = \frac{2^x + 1}{\sin x}$.

Solution: By the quotient rule we have

$$f'(x) = \frac{(2^x + 1)' \cdot \sin x - (2^x + 1) \cdot (\sin x)'}{(\sin x)^2} = \frac{2^x \ln 2 \cdot \sin x - (2^x + 1) \cdot \cos x}{(\sin x)^2}.$$

□

Example 62. Differentiate the function $f(x) = \frac{x^2 \ln x}{3x+4}$.

Solution: We need to use the combination of differentiation rules.





$$\begin{aligned}
f'(x) &= \frac{(x^2 \ln x)' (3x + 4) - x^2 \ln x \cdot (3x + 4)'}{(3x + 4)^2} \\
&= \frac{((x^2)' \ln x + x^2 (\ln x)') (3x + 4) - x^2 \ln x \cdot 3}{(3x + 4)^2} \\
&= \frac{(2x \ln x + x^2 \cdot \frac{1}{x}) (3x + 4) - x^2 \ln x \cdot 3}{(3x + 4)^2} \\
&= \frac{(2x \ln x + x) (3x + 4) - 3x^2 \ln x}{(3x + 4)^2}.
\end{aligned}$$

Apply the quotient rule.

Apply the product rule to differentiate the numerator.

Simplify.

□

Example 63. Differentiate the function $f(x) = \frac{2x - 7}{x^3 e^x}$.

Solution: We need to use the combination of differentiation rules.

$$\begin{aligned}
f'(x) &= \frac{(2x - 7)' x^3 e^x - (2x - 7) \cdot (x^3 e^x)'}{(x^3 e^x)^2} = \frac{2 \cdot x^3 e^x - (2x - 7) \cdot (3x^2 \cdot e^x + x^3 \cdot e^x)}{(x^3 e^x)^2} \\
&= \frac{x^2 e^x \cdot (2x - (2x - 7)(3 + x))}{x^6 \cdot (e^x)^2} = \frac{2x - 6x - 2x^2 + 21 + 7x}{x^4 \cdot e^x} = \frac{-2x^2 + 3x + 21}{x^4 e^x}.
\end{aligned}$$

□

Let f and g be differentiable functions. The relation

$$[f(g(x))]' = f'(g(x))g'(x)$$

holds whenever the right hand side is well defined. This formula is called the chain rule and we use it for differentiation of the composite functions.

Example 64. Differentiate the function $h(x) = (3x + 5)^4$.

Solution: To differentiate the function h , we begin by identifying its inside and outside functions. The given function is the composition of the outside function $f(x) = x^4$ and the inside function $g(x) = 3x + 5$. Their derivatives are $f'(x) = 4x^3$ and $g'(x) = 3$. We need to evaluate the derivative of the outer function at the inside function times the derivative of the inside function. We get

$$h'(x) = f'(g(x))g'(x) = 4(3x + 5)^3 \cdot 3 = 12(3x + 5)^3.$$

□

Example 65. Differentiate the function $h(x) = \sin(\sqrt{x})$.

Solution: The given function is the composition of $f(x) = \sin x$ and $g(x) = \sqrt{x} = x^{\frac{1}{2}}$.

$$f'(x) = \underbrace{\cos}_{\substack{\text{derivative} \\ \text{of the outside} \\ \text{function}}} \cdot \underbrace{(\sqrt{x})}_{\substack{\text{leave} \\ \text{the inside} \\ \text{function} \\ \text{alone}}} \cdot \underbrace{\left(x^{\frac{1}{2}}\right)'}_{\substack{\text{derivative} \\ \text{of the inside} \\ \text{function}}} = \cos(\sqrt{x}) \cdot \frac{1}{2} x^{-\frac{1}{2}} = \frac{\cos(\sqrt{x})}{2\sqrt{x}}.$$

□





Example 66. Differentiate the function $f(x) = x^4 e^{x^5}$.

Solution: The function f is the product of two functions: x^4 and e^{x^5} , so we will apply the product rule to the given function. Then we need to use the chain rule to the function e^{x^5} since it is a composite function.

$$\begin{aligned} f'(x) &= (x^4)' \cdot e^{x^5} + x^4 \cdot (e^{x^5})' = 4x^3 \cdot e^{x^5} + x^4 \cdot e^{x^5} \cdot (x^5)' \\ &= 4x^3 \cdot e^{x^5} + x^4 \cdot e^{x^5} \cdot 5x^4 = x^3 e^{x^5} (4 + 5x^5). \end{aligned}$$

□

Example 67. Differentiate the function $f(x) = \frac{\sin^2 x + 1}{\tan x}$.

Solution: We will apply the quotient rule to the given function f and the chain rule to the composite function $\sin^2 x$.

$$\begin{aligned} f'(x) &= \frac{(\sin^2 x + 1)' \cdot \tan x - (\sin^2 x + 1) \cdot (\tan x)'}{(\tan x)^2} = \frac{2 \sin x \cos x \cdot \tan x - (\sin^2 x + 1) \cdot \frac{1}{\cos^2 x}}{\tan^2 x} \\ &= \left(2 \sin x \cos x \cdot \frac{\sin x}{\cos x} - (\sin^2 x + 1) \cdot \frac{1}{\cos^2 x} \right) \cdot \frac{\cos^2 x}{\sin^2 x} = 2 \cos^2 x - 1 - \frac{1}{\sin^2 x}. \end{aligned}$$

□

3.4 Higher Order Derivatives

Example 68. Find the first four derivatives for the function $f(x) = \ln x + x^5$.

Solution: Using differentiation formulas and rules, we get

$$\begin{aligned} f'(x) &= \frac{1}{x} + 5x^4, & f''(x) &= (x^{-1})' + 5 \cdot 4x^3 = -x^{-2} + 20x^3, \\ f'''(x) &= 2x^{-3} + 60x^2, & f^{(4)}(x) &= -6x^{-4} + 120x. \end{aligned}$$

□

Example 69. On what intervals is the second derivative of the function $f(x) = \frac{1}{12}x^4 - 2x^2$ positive or negative?

Solution: First we find f' and f'' . We get

$$f'(x) = \frac{1}{3}x^3 - 4x \quad \text{and} \quad f''(x) = x^2 - 4.$$

The second derivative changes its sign when $f''(x) = 0$ or where f'' is not defined. Since f'' is defined for all $x \in \mathbb{R}$, we need only find where $f''(x) = 0$. We observe that

$$x^2 - 4 = 0 \quad \Rightarrow \quad (x - 2)(x + 2) = 0 \quad \Rightarrow \quad x_1 = -2, x_2 = 2.$$

The points x_1, x_2 divide the real line into three intervals. We need to pick a point from each region and plug it into the second derivative, then we draw the second derivative sign chart.





$$\begin{array}{ccccccc} & + & & - & & + & \\ & | & & | & & | & \\ \hline & -2 & & 2 & & & \end{array} \quad \text{sign } f''(x)$$

We conclude that the second derivative of f is positive on $(-\infty, -2) \cup (2, \infty)$ and negative on $(-2, 2)$. \square

3.5 Tangent and Normal lines

For the function f that is differentiable at the point x_0 , there exists the tangent line to the graph of f at x_0 and it is given by the equation

$$t: y = f(x_0) + f'(x_0) \cdot (x - x_0).$$

It is the line through the point $(x_0, f(x_0))$ with the slope $f'(x_0)$. The normal line is perpendicular to the tangent line at the point $(x_0, f(x_0))$ and it is given by the equation

$$n: y = f(x_0) - \frac{1}{f'(x_0)}(x - x_0).$$

If $f'(x_0) = 0$, then the equation of the normal line n is $x = x_0$.

Example 70. Find the equations of a tangent and a normal line to the function $f(x) = 3x^2 - 2\ln x$ at the point $T = (1, ?)$.

Solution: We have given function $f(x) = 3x^2 - 2\ln x$ and $x_0 = 1$. A tangent line intersects the graph at the point $T = (x_0, f(x_0))$. We know only the first coordinate $x_0 = 1$, so we need to find its the second coordinate. If we take the function value of the function f at $x_0 = 1$, we get

$$f(x_0) = f(1) = 3(1)^2 - 2 \cdot \ln 1 = 3,$$

so $T = (1, 3)$. Now, we find the derivative of the function f and its value at $x_0 = 1$.

$$f'(x) = (3x^2 - 2\ln x)' = 6x - \frac{2}{x} \Rightarrow f'(1) = 6 \cdot 1 - \frac{2}{1} = 4.$$

The equations of the tangent and normal lines are

$$\begin{array}{ll} t: & y = f(x_0) + f'(x_0) \cdot (x - x_0) \\ & y = 3 + 4 \cdot (x - 1) \\ & y = 4x - 1 \end{array} \quad \begin{array}{ll} n: & y = f(x_0) - \frac{1}{f'(x_0)} \cdot (x - x_0) \\ & y = 3 - \frac{1}{4} \cdot (x - 1) \\ & y = -\frac{1}{4}x + \frac{13}{4}. \end{array}$$

\square

Example 71. Find a point where the tangent line to the graph of $f(x) = 2x^3 + 3x^2 - 12x$ is horizontal.

Solution: A horizontal line has its slope equalled zero. So the slope of a horizontal tangent line is given as a solution of the equation $f'(x) = 0$.

$$f'(x) = 0 \Rightarrow 6x^2 + 6x - 12 = 0 \Rightarrow 6(x+2)(x-1) = 0 \Rightarrow x_1 = -2, x_2 = 1.$$

We conclude that the points where the tangent line is horizontal are $x = -2$ and $x = 1$. \square





3.6 Linear Approximations

We use the tangent line to the graph of the function f at the point $T = (x_0, f(x_0))$ to approximate f near $x = x_0$. So we can approximate

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

We call the tangent line **the linear approximation** of the function at $x = x_0$.

Example 72. Find the linear approximation to $f(x) = \sqrt[3]{x}$ at $x = 27$. Use this approximation to estimate the value of $\sqrt[3]{24}$.

Solution: We want to find the linear approximation which is the tangent line to the graph of the function f . For $x_0 = 27$ we have

$$f(x_0) = f(27) = \sqrt[3]{27} = 3.$$

Now, we find the derivative of the function f and its value at $x_0 = 27$.

$$f'(x) = (\sqrt[3]{x})' = \left(x^{\frac{1}{3}}\right)' = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}} \Rightarrow f'(27) = \frac{1}{3\sqrt[3]{27^2}} = \frac{1}{27}.$$

The linear approximation is

$$t(x) = f(x_0) + f'(x_0)(x - x_0) = 3 + \frac{1}{27}(x - 27) = \frac{1}{27}x + 2.$$

Using this approximation we can estimate $\sqrt[3]{24}$ as

$$\sqrt[3]{24} = f(24) \approx t(24) = \frac{1}{27} \cdot 24 + 2 = \frac{8}{9} + 2 \doteq 2.8889.$$

This value is very close to the exact value of $\sqrt[3]{24}$, which can be found by using a calculator to four decimal places as 2.8845.

□

Example 73. Approximate (without using calculator) the value of $\frac{4}{1.05}$.

Solution: Let $f(x) = \frac{4}{x}$. We want to find $f(1.05)$, so we need a point x_0 that is close to $x = 1.05$ and also is easy to put into the function f . It is obviously $x_0 = 1$. Now, we find the derivative of the function f and its value at $x_0 = 1$.

$$f'(x) = (4x^{-1})' = -4x^{-2} = -\frac{4}{x^2} \Rightarrow f'(1) = -\frac{4}{1} = -4.$$

Using the linear approximation, we get

$$\begin{aligned} f(x) &\approx f(x_0) + f'(x_0)(x - x_0) \\ f(1.05) &\approx f(1) + f'(1)(1.05 - 1) \\ \frac{4}{1.05} &\approx 4 - 4 \cdot 0.05 \\ \frac{4}{1.05} &\approx 3.8. \end{aligned}$$

The exact value of $\frac{4}{1.05}$ to three decimal places is 3.809.

□





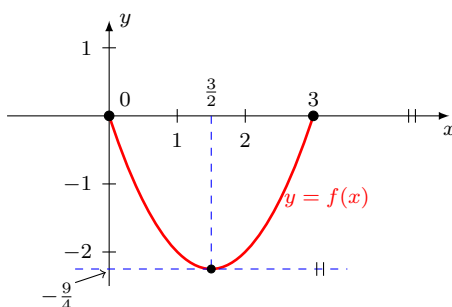
3.7 The Mean Value Theorem

We will look at the meaning and some applications of Rolle's Theorem and the Mean Value Theorem.

Example 74. Verify that the function $f(x) = x^2 - 3x$ satisfies the conditions stated in Rolle's Theorem and find all values x_0 in the interval $[0, 3]$ where $f'(x_0) = 0$.

Solution: The given function f is continuous and differentiable for all $x \in \mathbb{R}$. Moreover, we can see that $f(0) = f(3) = 0$. The conditions of Rolle's Theorem are satisfied, so there exists a point $x_0 \in (0, 3)$ such that $f'(x_0) = 0$. It is easy to find that

$$f'(x) = 0 \Rightarrow 2x - 3 = 0 \Rightarrow x_0 = \frac{3}{2}.$$



□

Example 75. Verify that the function $f(x) = -5x^2 + 20$ over the interval $[0, 2]$ satisfies the conditions stated in the Mean Value Theorem and find all values c guaranteed by the Mean Value Theorem.

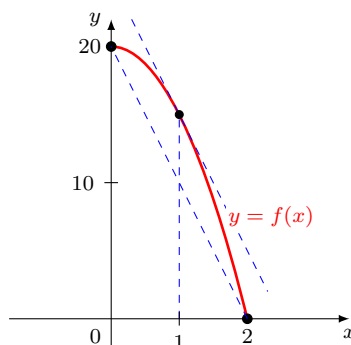
Solution: The function f is continuous on the closed interval $[0, 2]$ and differentiable on the interval's interior $(0, 2)$. So, the function f satisfies the assumptions of the Mean Value Theorem and there must be a point c where

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{0 - 20}{2 - 0} = -10.$$

The derivative of f is $f'(x) = -10x$, thus we want to find c where

$$-10c = -10.$$

Solving this equation for c , we obtain $c = 1$. At this point, the tangent line is parallel to the secant line connecting the two endpoints $(0, 20)$ and $(2, 0)$ of the graph of f . Their slopes are equal and are exactly the value $f'(c) = -10$.



□





3.8 L'Hospital's Rule

In this section we will evaluate limits using l'Hospital's Rule. This rule can be applied to the two indeterminate forms $0/0$ and ∞/∞ under the following conditions. Let $x_0 \in \mathbb{R}^*$ and let f and g be functions defined and differentiable in some ring neighbourhood of the point x_0 . Suppose that either

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = 0 \quad \text{or} \quad \lim_{x \rightarrow x_0} |g(x)| = \infty$$

holds. Then

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$$

holds if the limit on the right-hand side exists (finite or infinite). We must differentiate the numerator and differentiate the denominator and then evaluate the limit. If this new limit is again one of the indeterminate forms, we can repeat l'Hospital's rule.

Example 76. Evaluate the limit $\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - 7x + 10}$.

Solution: Here $f(x) = x^2 + x - 6$, $g(x) = x^2 - 7x + 10$ and $x_0 = 2$. The functions f and g are continuous and differentiable on all neighbourhoods of the point x_0 . Further, the numerator $f \rightarrow 0$ and the denominator $g \rightarrow 0$. So we can apply l'Hospital's Rule to evaluate given limit. We divide the derivative of f by the derivative of g .

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - 7x + 10} = \left[\frac{0}{0} \right] = \lim_{x \rightarrow 2} \frac{(x^2 + x - 6)'}{(x^2 - 7x + 10)'} = \lim_{x \rightarrow 2} \frac{2x + 1}{2x - 7} = \frac{2 \cdot 2 + 1}{2 \cdot 2 - 7} = -\frac{5}{3}.$$

□

Example 77. Evaluate the limit $\lim_{x \rightarrow 0} \frac{\cos 3x - \cos 7x}{x^2}$.

Solution: As $x \rightarrow 0$, both the numerator and denominator approach zero. Therefore we can calculate the limit using l'Hospital's Rule.

$$\lim_{x \rightarrow 0} \frac{\cos 3x - \cos 7x}{x^2} = \left[\frac{0}{0} \right] \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{(\cos 3x - \cos 7x)'}{(x^2)'} = \lim_{x \rightarrow 0} \frac{-3 \sin 3x + 7 \sin 7x}{2x}.$$

This new limit is also a $0/0$ indeterminate form. We can continue to differentiate.

$$\lim_{x \rightarrow 0} \frac{-3 \sin 3x + 7 \sin 7x}{2x} = \left[\frac{0}{0} \right] \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{-9 \cos 3x + 49 \cos 7x}{2} = \frac{-9 + 49}{2} = 20.$$

We conclude that

$$\lim_{x \rightarrow 0} \frac{\cos 3x - \cos 7x}{x^2} = 20.$$

□





Example 78. Evaluate the limit $\lim_{x \rightarrow \infty} \frac{4x^4 - 2x^2 + 2x}{-2x^4 + 3x - 1}$.

Solution: We evaluated this limit in the example 47, the answer is -2 . Now we apply l'Hospital's Rule.

$$\lim_{x \rightarrow \infty} \frac{4x^4 - 2x^2 + 2x}{-2x^4 + 3x - 1} = \left[\frac{\infty}{\infty} \right] \stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{16x^3 - 4x + 2}{-8x^3 + 3} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{48x^2 - 4}{-24x^2} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{96x}{-48x} = -2.$$

□

Example 79. Evaluate the limit $\lim_{x \rightarrow 0^+} \frac{\ln x}{\cot x}$.

Solution: We can see that $\lim_{x \rightarrow 0^+} \ln x = -\infty$ and $\lim_{x \rightarrow 0^+} \cot x = \infty$. We can apply l'Hospital's Rule.

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{\cot x} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{\sin^2 x}} = - \lim_{x \rightarrow 0^+} \frac{\sin^2 x}{x} = \left[\frac{0}{0} \right] \stackrel{\text{L'H}}{=} - \lim_{x \rightarrow 0^+} \frac{2 \sin x \cos x}{1} = -\frac{0}{1} = 0.$$

□

To evaluate limits involving the indeterminate forms such as $[0 \cdot \infty]$, $[\infty - \infty]$, $[0^0]$, $[\infty^0]$, $[1^\infty]$, we must convert these forms to a $0/0$ or ∞/∞ form and try to use l'Hospital's Rule to them.

Example 80. Evaluate the limit $\lim_{x \rightarrow 0^+} x \cdot e^{\frac{1}{x}}$.

Solution: For $x \rightarrow 0^+$ we get the indeterminate form $0 \cdot \infty$. We need to rewrite the expression $x \cdot e^{\frac{1}{x}}$ as a quotient.

$$\lim_{x \rightarrow 0^+} x \cdot e^{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{e^{\frac{1}{x}}}{\frac{1}{x}} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} \frac{e^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} e^{\frac{1}{x}} = \infty.$$

□

Example 81. Evaluate the limit $\lim_{x \rightarrow \infty} (x - x \ln x)$.

Solution: For $x \rightarrow \infty$ we get the indeterminate form $\infty - \infty$. We need to rewrite the expression $x - x \ln x$.

$$\lim_{x \rightarrow \infty} (x - x \ln x) = \lim_{x \rightarrow \infty} x \cdot (1 - \ln x).$$

This limit leads to $\infty \cdot (-\infty)$, which is not an indeterminate form. We conclude that

$$\lim_{x \rightarrow \infty} (x - x \ln x) = -\infty.$$

□

Example 82. Evaluate the limit $\lim_{x \rightarrow \infty} x^{\frac{1}{\sqrt{x}}}$.

Solution: This is the indeterminate form ∞^0 . At first we rewrite the given expression.

$$x^{\frac{1}{\sqrt{x}}} = e^{\frac{1}{\sqrt{x}} \cdot \ln x} = e^{\frac{\ln x}{\sqrt{x}}}.$$





Let's use L'Hospital's Rule on the power.

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = \left[\frac{\infty}{\infty} \right] \stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{2\sqrt{x}}{x} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0.$$

Since the exponential function is continuous, we conclude that

$$\lim_{x \rightarrow \infty} x^{\frac{1}{\sqrt{x}}} = \lim_{x \rightarrow \infty} e^{\frac{\ln x}{\sqrt{x}}} = e^{\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}}} = e^0 = 1.$$

□

3.9 Monotonicity of the Function and Local Extrema

Using the derivative of a function we will find the intervals of monotonicity of the functions and their minimum and maximum values in the following steps.

1. Find intervals of the domain of f .
2. Find the derivative f' . Find critical points, it means points, where $f'(x) = 0$ or f' does not exist.
3. Mark points from Step 2 on the real line. These points divide the real line into several subintervals. Determine the sign of the derivative on each subinterval. We substitute convenient points from each subinterval to f' to find the sign.
4. Determine monotonicity from the signs of f' . The function f is increasing on intervals where f' is positive. It is decreasing on intervals where f' is negative.
5. Determine local extrema. If f changes from decreasing to increasing and is continuous at the point x_0 , then there is a local minimum at the point x_0 . If f changes from increasing to decreasing and is continuous at the point x_0 , then there is a local maximum at the point x_0 .

Example 83. Find local extrema and intervals of monotonicity of the function $f(x) = \frac{x^3}{3} + \frac{x^2}{2} - 6x + 1$.

Solution: Function f is defined for $x \in (-\infty, \infty)$. We find the derivative

$$f'(x) = x^2 + x - 6, \quad x \in \mathbb{R}.$$

There are no points where f' does not exist. We solve the equation $f'(x) = 0$ to find critical points.

$$x^2 + x - 6 = 0 \quad \Rightarrow \quad (x+3)(x-2) = 0 \quad \Rightarrow \quad x_1 = -3, \quad x_2 = 2.$$

These points divide the number line into three intervals

$$(-\infty, -3), \quad (-3, 2), \quad (2, \infty).$$

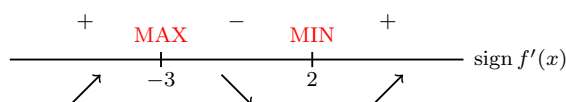
We will pick test points from each region, for example $x = -4$, $x = 0$, $x = 3$, to see if the derivative is positive or negative at these points and thus positive or negative in each region.



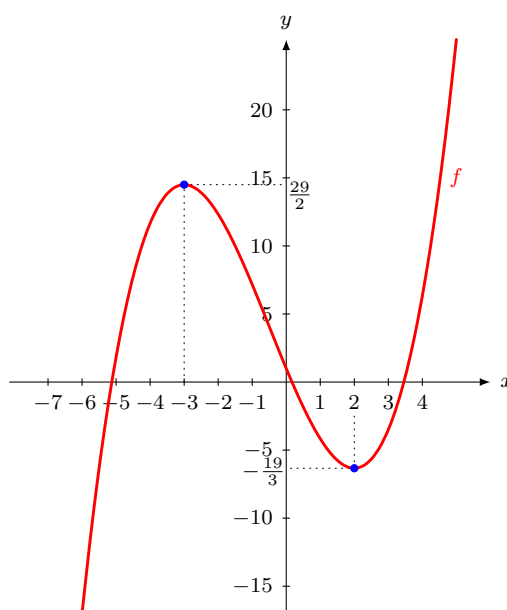


$$\begin{aligned} x \in (-\infty, -3): & \quad f'(x) > 0, \\ x \in (-3, 2): & \quad f'(x) < 0, \\ x \in (2, \infty): & \quad f'(x) > 0. \end{aligned}$$

We have the sign of f' in each interval and we can draw the following scheme.



The function f is increasing on $(-\infty, -3]$ and on $[2, \infty)$ and decreasing on $[-3, 2]$. The value $f(-3) = 29/2$ is a local maximum and the value $f(2) = -19/3$ is a local minimum. The following graph confirms our results.



□

Example 84. Find local extrema and intervals of monotonicity of the function $f(x) = \frac{\sqrt{x}}{4} + \frac{1}{x}$.

Solution: Function f is defined for $x \geq 0$. The derivative is

$$f'(x) = \left(\frac{1}{4} \cdot x^{\frac{1}{2}} + x^{-1} \right)' = \frac{1}{4} \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}} - x^{-2} = \frac{1}{8\sqrt{x}} - \frac{1}{x^2} = \frac{x^{3/2} - 8}{8x^2}.$$

The derivative f' is defined for $x > 0$. We solve the equation $f'(x) = 0$ to find critical points.

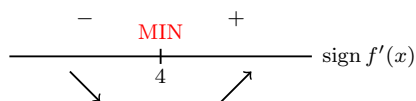
$$\frac{x^{3/2} - 8}{8x^2} = 0 \Rightarrow x^{3/2} - 8 = 0 \Rightarrow x^{3/2} = 8 \Rightarrow x^{1/2} = 2 \Rightarrow x = 4.$$

We have one stationary point $x = 4$. This point divides the interval $(0, \infty)$ into two intervals $(0, 4)$ and $(4, \infty)$. Let's consider test points $c_1 = 1$, $c_2 = 16$ from these intervals. The derivative at these points is $f'(1) = -7/8$ and $f'(16) = 7/256$. We obtain the following scheme.

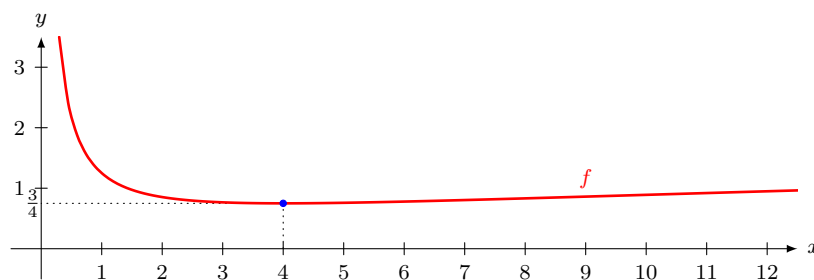


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Function f is decreasing on $(0, 4]$ and increasing on $[4, \infty)$. Critical point $x_0 = 4$ was candidate for a local extreme. The value $f(4) = \frac{3}{4}$ is a local minimum. The following graph confirms our result.



□

3.10 Global Extrema

If a function is continuous at every point of a **closed**, bounded interval $[a, b]$, it has a global (in other words absolute) maximum and a global minimum value over this interval. It is guaranteed by Extreme Value Theorem. We can find global extrema in the following steps.

1. Verify that the function is continuous on the interval $[a, b]$.
2. Find the critical points of f that are in the interval $[a, b]$.
3. Evaluate the function f at the calculated critical points and the endpoints a, b .
4. Choose a global maximum and minimum from among these values. Global maximum is the largest value that the function attains and global minimum is the smallest value.

Example 85. Find global extrema of the function $f(x) = (x - \frac{1}{2}x^2)^2$ for $x \in [-1, 1]$.

Solution: We will follow the procedure given above. The function f is continuous for all $x \in \mathbb{R}$ and in particular is continuous on the given interval. We differentiate the function f to find its critical points.

$$f'(x) = 2 \cdot (x - \frac{1}{2}x^2) \cdot (1 - x) = 2 \cdot x \cdot (1 - \frac{1}{2}x) \cdot (1 - x) = 0 \quad \Rightarrow \quad x_1 = 0, \quad x_2 = 1, \quad x_3 = 2.$$

Critical point $x_1 = 0$ lies in the interval $[-1, 1]$, $x_2 = 1$ is its endpoint and the point $x = 2$ is not in the interval of interest (We ignore it.). We evaluate the function f at the critical point $x_1 = 0$ and at the endpoints of the interval $[-1, 1]$.

$$f(-1) = \frac{9}{4}, \quad f(0) = 0, \quad f(1) = \frac{1}{4}.$$

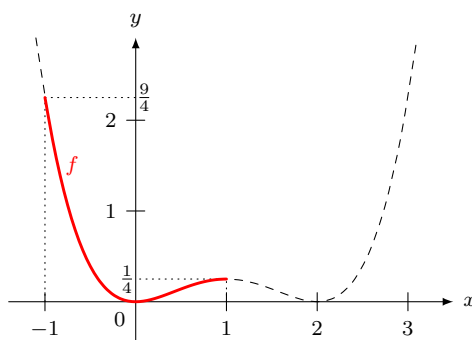


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Global extrema are the largest and smallest values of the function. The global maximum of the f is $\frac{9}{4}$ and it occurs at $x = -1$ (an endpoint). The global minimum of the f is 0 and it occurs at $x = 0$ (a critical point). The situation is shown in the following graph.



□

Example 86. Find global extrema of the function $f(x) = \sin x + \cos x$ for $x \in [0, 2\pi]$.

Solution: The function f is continuous for all $x \in \mathbb{R}$ and in particular is continuous on the given interval. We differentiate the function f to find its critical points.

$$f'(x) = \cos x - \sin x = 0 \quad \Rightarrow \quad \sin x = \cos x.$$

To solve this equation we continue as follows.

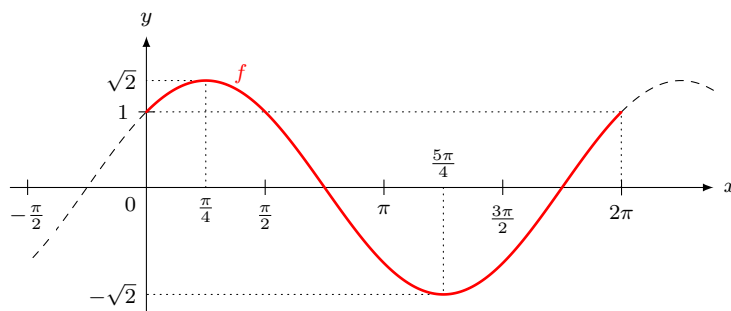
$$\begin{aligned} \sin x &= \cos x \\ \frac{\sin x}{\cos x} &= 1 \\ \tan x &= 1 \\ \tan x &= \tan\left(\frac{\pi}{4} + k\pi\right), \quad k \in \mathbb{Z} \\ x &= \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}. \end{aligned}$$

The only solutions that lie in the interval $[0, 2\pi]$ are $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$. These points are critical points of the function f lying in the interval $[0, 2\pi]$. We evaluate the function f at these points and at the endpoints of the interval $[0, 2\pi]$.

$$f(0) = 1, \quad f\left(\frac{\pi}{4}\right) = \sqrt{2}, \quad f\left(\frac{5\pi}{4}\right) = -\sqrt{2}, \quad f(2\pi) = 1.$$

The global maximum of the f is $\sqrt{2}$ and it occurs at $x = \frac{\pi}{4}$. The global minimum of the f is $-\sqrt{2}$ and it occurs at $x = \frac{5\pi}{4}$. The situation is shown in the following graph.





□

3.11 Concavity

We will use the second derivative of a function to determine the shape of its graph. The graph of the function f can curve upward or curve downward, we say that the function f is concave up or concave down, respectively.

Let a function f be twice differentiable on an interval I . Then f is concave up on I if and only if $f''(x) > 0$ for all $x \in I$ and f is concave down on I if and only if $f''(x) < 0$ for all $x \in I$. The point x_0 in which the type of concavity changes is said to be a point of inflection of the function f .

Example 87. Sketch a graph of a function f that is:

- a) decreasing, concave down on $(0, 1)$,
- b) decreasing, concave up on $(1, 2)$,
- c) increasing, concave up on $(2, 3)$ and
- d) decreasing, concave up on $(3, 4)$.

□

Example 88. Investigate concavity and inflection points of the function $f(x) = x + x^2 - x^3$.

Solution: The function f is continuous and differentiable for all $x \in \mathbb{R}$. We have

$$f'(x) = 1 + 2x - 3x^2, \quad f''(x) = 2 - 6x.$$

The second derivative is always defined. We solve the equation $f''(x) = 0$ to find the possible inflection points.

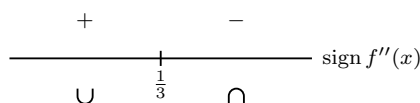
$$2 - 6x = 0 \quad \Rightarrow \quad x = \frac{1}{3}.$$

This point divides the number line into two intervals $(-\infty, \frac{1}{3})$ and $(\frac{1}{3}, \infty)$. We need to pick a point from each region and plug it into the second derivative to determine the sign of f'' and thus the concavity of f . Let's consider test points $c_1 = 0$, $c_2 = 1$ from these intervals. The second derivative at these points is $f''(0) = 2$ and $f''(1) = -4$. We can draw the following scheme.

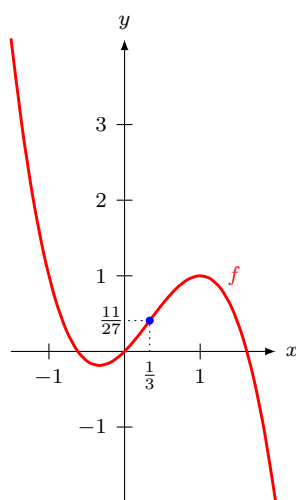


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We have got the following intervals of concavity, the function f is concave down on $(\frac{1}{3}, \infty)$, it is concave up on $(-\infty, \frac{1}{3})$. The function f has one point of inflection $x = \frac{1}{3}$. The situation is shown in the following graph.



□

Example 89. Investigate concavity and inflection points of the function $f(x) = \frac{x}{x^2 - 4}$.

Solution: The function f is defined for all $x \in (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$. We need to find f' and f'' .

$$\begin{aligned} f'(x) &= \frac{x' \cdot (x^2 - 4) - x \cdot (x^2 - 4)'}{(x^2 - 4)^2} = \frac{1 \cdot (x^2 - 4) - x \cdot 2x}{(x^2 - 4)^2} = -\frac{x^2 + 4}{(x^2 - 4)^2}, \\ f''(x) &= -\frac{(x^2 + 4)' \cdot (x^2 - 4)^2 - (x^2 + 4) \cdot ((x^2 - 4)^2)'}{(x^2 - 4)^4} = \\ &= -\frac{2x \cdot (x^2 - 4)^2 - (x^2 + 4) \cdot 2(x^2 - 4) \cdot 2x}{(x^2 - 4)^4} = -\frac{2x \cdot (x^2 - 4) - (x^2 + 4) \cdot 4x}{(x^2 - 4)^3} = \\ &= -\frac{2x^3 - 8x - 4x^3 - 16x}{(x^2 - 4)^3} = -\frac{-2x^3 - 24x}{(x^2 - 4)^3} = \frac{2x \cdot (x^2 + 12)}{(x^2 - 4)^3}. \end{aligned}$$

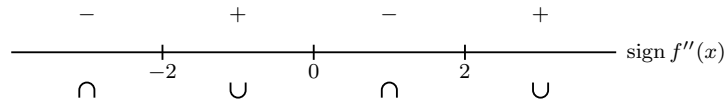
We solve the equation $f''(x) = 0$ to find the possible inflection points.

$$\frac{2x \cdot (x^2 + 12)}{(x^2 - 4)^3} = 0 \Rightarrow 2x \cdot (x^2 + 12) = 0 \Rightarrow x = 0.$$





The second derivative f'' is not defined when $x = \pm 2$, since the denominator of f'' is 0 at these points. Moreover, given function f is not defined at $x = \pm 2$, so there is only one possible point of inflection $x = 0$. This point and the points $x = \pm 2$ divide the real line into four intervals $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, $(2, \infty)$, on which f'' is either positive or negative. We determine the sign of f'' by evaluating f'' at a convenient point in each interval.



For selected $c = -10 \in (-\infty, -2)$ the denominator is positive and the numerator is negative, so $f''(c)$ is negative. We conclude that f is concave down on $(-\infty, -2)$. For $c = -1 \in (-2, 0)$ the denominator is negative and the numerator is negative and $f''(c)$ is thus positive. Hence, f is concave up on $(-2, 0)$. For any number $c \in (0, 2)$ the second derivative f'' is negative and f is concave down on this interval. For any number $c \in (2, \infty)$ the second derivative f'' is positive and f is concave up on this interval. \square

3.12 Asymptotes

Example 90. Find all asymptotes of the function $f(x) = \frac{2x^2 - 2x - 4}{x^2 + x - 6}$.

Solution: The domain for this function are all the values x for which we do not have division by zero. We need set the denominator equal to zero and solve.

$$x^2 + x - 6 = 0 \Rightarrow x_{1,2} = \frac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-6)}}{2} = \frac{-1 \pm 5}{2} \Rightarrow x_1 = -3, x_2 = 2.$$

We will get division by zero if we plug in $x = -3$ or $x = 2$. So, the domain is $D_f = (-\infty, -3) \cup (-3, 2) \cup (2, \infty)$. The points $x = -3$ and $x = 2$ are candidates for a vertical asymptote. Since

$$\lim_{x \rightarrow -3^-} \frac{2x^2 - 2x - 4}{x^2 + x - 6} = \left[\frac{20}{0^+} \right] = +\infty,$$

$$\lim_{x \rightarrow 2} \frac{2x^2 - 2x - 4}{x^2 + x - 6} = \left[\frac{0}{0} \right] \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 2} \frac{4x - 2}{2x + 1} = \frac{8 - 2}{4 + 1} = \frac{6}{5},$$

the function f has a vertical asymptote $x = -3$. There is no vertical asymptote at 2. Now, we test the existence of the oblique asymptotes.

$$k = \lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \frac{\frac{2x^2 - 2x - 4}{x^2 + x - 6}}{x} = \lim_{x \rightarrow +\infty} \frac{2x^2 - 2x - 4}{x^3 + x^2 - 6x} = \lim_{x \rightarrow +\infty} \frac{2x^2}{x^3} = 0,$$

$$q = \lim_{x \rightarrow +\infty} [f(x) - kx] = \lim_{x \rightarrow +\infty} \left[\frac{2x^2 - 2x - 4}{x^2 + x - 6} - 0 \cdot x \right] =$$

$$= \lim_{x \rightarrow +\infty} \frac{2x^2 - 2x - 4}{x^2 + x - 6} = \lim_{x \rightarrow +\infty} \frac{2x^2}{x^2} = 2.$$

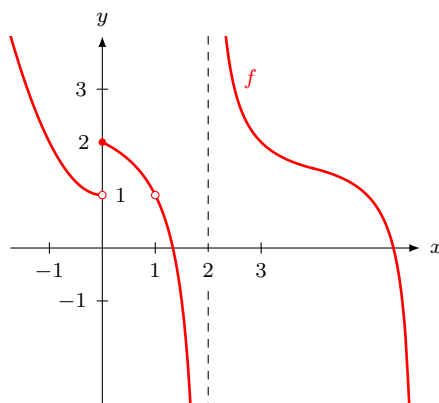




The same computations can be extended also to the case of limits at $-\infty$. We conclude that the line $y = 2$ is the asymptote at infinity and also minus infinity. It is the horizontal asymptote of the graph of the function f .

□

Example 91. Use the graph of f



to determine each of the following values.

a) $\lim_{x \rightarrow 0^-} f(x)$, $\lim_{x \rightarrow 0^+} f(x)$, $\lim_{x \rightarrow 0} f(x)$, $f(0)$,

b) $\lim_{x \rightarrow 1^-} f(x)$, $\lim_{x \rightarrow 1^+} f(x)$, $\lim_{x \rightarrow 1} f(x)$, $f(1)$,

c) $\lim_{x \rightarrow 2^-} f(x)$, $\lim_{x \rightarrow 2^+} f(x)$, $\lim_{x \rightarrow 2} f(x)$, $f(2)$.

Solution: We attain the following values.

a) $\lim_{x \rightarrow 0^-} f(x) = 1$, $\lim_{x \rightarrow 0^+} f(x) = 2$, $\lim_{x \rightarrow 0} f(x)$ does not exist, $f(0) = 2$,

b) $\lim_{x \rightarrow 1^-} f(x) = 1$, $\lim_{x \rightarrow 1^+} f(x) = 1$, $\lim_{x \rightarrow 1} f(x) = 1$, $f(1)$ is undefined,

c) $\lim_{x \rightarrow 2^-} f(x) = -\infty$, $\lim_{x \rightarrow 2^+} f(x) = \infty$, $\lim_{x \rightarrow 2} f(x)$ does not exist, $f(2)$ is undefined.

□



4 Indefinite Integral

For given function f defined on an interval I we want to find the function F on I which satisfies $F'(x) = f(x)$ for all x from the interior of I . Such defined function F is called an antiderivative of the function f . The set of all antiderivatives of f is the indefinite integral of f .

4.1 Indefinite Integral

The following table lists the common Indefinite Integral Rules.

$\int 0 \, dx = C$	$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$
$\int 1 \, dx = x + C$	$\int \frac{1}{\sqrt{1-x^2}} \, dx = -\arccos x + C$
$\int k \, dx = k \cdot x + C \quad (k \in \mathbb{R})$	$\int \frac{1}{1+x^2} \, dx = \operatorname{arctg} x + C$
$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	$\int \frac{1}{1+x^2} \, dx = -\operatorname{arccotg} x + C$
$\int \frac{1}{x} \, dx = \ln x + C$	$\int \frac{1}{\sqrt{x^2+a}} \, dx = \ln \left(x + \sqrt{x^2+a} \right) + C \quad (a \neq 0)$
$\int e^x \, dx = e^x + C$	$\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \arcsin \frac{x}{a} + C$
$\int a^x \, dx = \frac{a^x}{\ln a} + C$	$\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a} \cdot \operatorname{arctg} \frac{x}{a} + C$
$\int \sin x \, dx = -\cos x + C$	$\int \frac{f'(x)}{f(x)} \, dx = \ln f(x) + C$
$\int \cos x \, dx = \sin x + C$	$\int c f(x) \, dx = c \int f(x) \, dx$
$\int \frac{1}{\cos^2 x} \, dx = \operatorname{tg} x + C$	$\int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx$
$\int \frac{1}{\sin^2 x} \, dx = -\operatorname{cotg} x + C$	

Example 92. Evaluate the integral $\int \left(4 \cdot 3^x - \sqrt[3]{x} + \frac{2}{x^3} \right) dx$.



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Solution: We apply the linearity rule and integrate the individual terms using integration formulas.

$$\begin{aligned} \int \left(4 \cdot 3^x - \sqrt[3]{x} + \frac{2}{x^3} \right) dx &= 4 \int 3^x dx - \int x^{\frac{1}{3}} dx + 2 \int x^{-3} dx = 4 \frac{3^x}{\ln 3} - \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + 2 \frac{x^{-2}}{-2} = \\ &= \frac{4 \cdot 3^x}{\ln 3} - \frac{3\sqrt[3]{x^4}}{4} - \frac{1}{x^2} + C. \end{aligned}$$

□

Example 93. Evaluate the integral $\int \frac{1 + \cos 2x}{1 - \cos 2x} dx$.

Solution: We need to rewrite the integrand using trig identities.

$$\begin{aligned} \int \frac{1 + \cos 2x}{1 - \cos 2x} dx &= \int \frac{\sin^2 x + \cos^2 x + \cos^2 x - \sin^2 x}{\sin^2 x + \cos^2 x - \cos^2 x + \sin^2 x} dx = \int \frac{2 \cos^2 x}{2 \sin^2 x} dx = \\ &= \int \frac{1 - \sin^2 x}{\sin^2 x} dx = \int \left(\frac{1}{\sin^2 x} - 1 \right) dx = -\cot x - x + C. \end{aligned}$$

□

4.2 Integration by Parts

Let functions u and v have continuous first derivatives on an open interval I . Then the integration-by-parts formula for these two functions says

$$\int u(x)v'(x) dx = u(x)v(x) - \int u'(x)v(x) dx.$$

Example 94. Evaluate the integral $\int x \cos x dx$.

Solution: We need to express the integrated function as a product of two functions denoted $u(x)$ and $v'(x)$. In our case we have $u(x) = x$, $v'(x) = \cos x$. We determine $u'(x)$ by differentiating the expression $u(x)$ and $v(x)$ by integrating $v'(x)$, no constant of integration is used.

$$u'(x) = (x)' = 1, \quad v(x) = \int \cos x dx = \sin x.$$

We substitute all these expressions into the integration-by-parts formula, giving

$$\int x \cdot \cos x dx = x \cdot (\sin x) - \int 1 \cdot (\sin x) dx.$$

Now, we only evaluate the integral of $\sin x$.

$$\int x \cdot \cos x dx = x \sin x + \cos x + C.$$

□





Example 95. Evaluate the integral $\int (3x^2 - x + 2)e^x dx$.

Solution: We use the integration-by-parts formula for $u = 3x^2 - x + 2$, $v' = e^x$. We usually write the integration by parts into a table.

$$\int (3x^2 - x + 2)e^x dx = \left| \begin{array}{ll} u = 3x^2 - x + 2 & v' = e^x \\ u' = 6x - 1 & v = e^x \end{array} \right| = (3x^2 - x + 2)e^x - \int (6x - 1)e^x dx.$$

The new integral on the right is similar to the initial integral and important is that it is simpler than the one we started with. We can do integration by parts again with $u = 6x - 1$ and $v' = e^x$.

$$\begin{aligned} (3x^2 - x + 2)e^x - \int (6x - 1)e^x dx &= \left| \begin{array}{ll} u = 6x - 1 & v' = e^x \\ u' = 6 & v = e^x \end{array} \right| = \\ &= (3x^2 - x + 2)e^x - \left((6x - 1)e^x - \int 6e^x dx \right). \end{aligned}$$

The last step is to evaluate integral of $6e^x$ and simplify. The answer is

$$\int (3x^2 - x + 2)e^x dx = (3x^2 - x + 2)e^x - (6x - 1)e^x + 6e^x + C.$$

□

Example 96. Evaluate the integral $\int \frac{\ln x}{x^4} dx$.

Solution: Rewriting the integral leads to

$$\int x^{-4} \cdot \ln x dx$$

and we can integrate by parts with $u = \ln x$ and $v' = x^{-4}$.

$$\begin{aligned} \int x^{-4} \cdot \ln x dx &= \left| \begin{array}{ll} u = \ln x & v' = x^{-4} \\ u' = x^{-1} & v = \frac{x^{-3}}{-3} \end{array} \right| = \frac{x^{-3}}{-3} \cdot \ln x - \int x^{-1} \cdot \frac{x^{-3}}{-3} dx = \\ &= \frac{x^{-3}}{-3} \cdot \ln x + \frac{1}{3} \int x^{-4} dx = \frac{x^{-3}}{-3} \cdot \ln x + \frac{1}{3} \cdot \frac{x^{-3}}{-3} = -\frac{\ln x}{3x^3} - \frac{1}{9x^3} + C. \end{aligned}$$

□

4.3 First Substitution Rule

Let $t = g(x)$ be a differentiable function on J whose range is an interval I , and f be continuous on I , then

$$\int f(g(x)) g'(x) dx = \left| \begin{array}{l} t = g(x) \\ dt = g'(x) dx \end{array} \right| = \int f(t) dt = F(t) = F(g(x)) \quad \text{on } J.$$



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Example 97. Evaluate the integral $\int 3 \cos(3x - 5) dx$.

Solution: Let us put $t = 3x - 5$ and compute the differential

$$dt = 3 dx.$$

We write t instead of $3x - 5$ and dt instead of $3 dx$ in the integral.

$$\int 3 \cos(3x - 5) dx = \left| \begin{matrix} t = 3x - 5 \\ dt = 3 dx \end{matrix} \right| = \int \cos t dt = \sin t = \sin(3x - 5) + C.$$

Don't forget to return back to the original variable. □

Example 98. Evaluate the integral $\int \frac{\sin 2x}{\cos^5 2x} dx$.

Solution: Let us consider $t = \cos 2x$, it leads to $dt = -2 \sin 2x dx$. The integrand has a $\sin 2x dx$ term but not a $-2 \sin 2x dx$. We can introduce the factor of -2 after the integral sign if we compensate for it by a factor of $-1/2$ in front of the integral sign.

$$\begin{aligned} \int \frac{\sin 2x}{\cos^5 2x} dx &= -\frac{1}{2} \int \frac{-2 \sin 2x}{\cos^5 2x} dx = \left| \begin{matrix} t = \cos 2x \\ dt = -2 \sin 2x dx \end{matrix} \right| = -\frac{1}{2} \int \frac{1}{t^5} dt = \\ &= -\frac{1}{2} \int t^{-5} dt = -\frac{1}{2} \cdot \frac{t^{-4}}{-4} = \frac{1}{8 \cos^4 2x} + C. \end{aligned}$$

□

Example 99. Evaluate the integral $\int x \sqrt{x-4} dx$.

Solution: Let's make the substitution $t = x - 4$. For this t we have $dt = dx$. There is x in the integrand, so we need to express x in terms of t . Since $t = x - 4$, we can replace x with $t + 4$. Then

$$\begin{aligned} \int x \sqrt{x-4} dx &= \int (t+4) \sqrt{t} dt = \int (t+4) t^{\frac{1}{2}} dt = \int \left(t^{\frac{3}{2}} + 4t^{\frac{1}{2}} \right) dt = \\ &= \frac{t^{\frac{5}{2}}}{\frac{5}{2}} + 4 \frac{t^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2\sqrt{(x-4)^5}}{5} + \frac{8\sqrt{(x-4)^3}}{3} + C. \end{aligned}$$

□

4.4 Integration of Rational Functions

A rational function is by definition the quotient of two polynomials

$$R(x) = \frac{P_n(x)}{Q_m(x)},$$

where $P_n(x)$ is an n -degree polynomial and $Q_m(x)$ is an m -degree polynomial. If the degree of the denominator is less than the degree of the numerator, such function is called improper and we perform





polynomial long division with remainder to rewrite improper rational function as a sum of a polynomial and a proper rational function with the same denominator.

Let $R(x) = \frac{P_n(x)}{Q_m(x)}$ be a proper rational function. Suppose that the polynomials $P_n(x)$ and $Q_m(x)$ have no common zeros.

- Let us assign to each simple real root x_0 of the polynomial $Q_m(x)$ the fraction

$$\frac{A}{x - x_0},$$

where A is some (not yet determined) real constant.

- Let us assign to each real root x_0 of the multiplicity k of the polynomial $Q_m(x)$ the k -tuple of the fractions

$$\frac{A_1}{x - x_0}, \quad \frac{A_2}{(x - x_0)^2}, \quad \dots, \quad \frac{A_k}{(x - x_0)^k},$$

where A_i are some real constants.

- An irreducible quadratic factor $(x^2 + Mx + N)$ of the polynomial $Q_m(x)$ has a pair of the mutually conjugate complex roots. Let us assign to this pair of complex roots the fraction

$$\frac{Bx + C}{x^2 + Mx + N},$$

where B and C are some (not yet determined) real numbers.

In the following two examples we will show two methods of the partial fraction decomposition. More demonstration examples can be found in the lecture presentation slides.

Example 100. Evaluate the integral $\int \frac{10 - x}{(x - 1)(x + 2)} dx$.

Solution: The first step is to factor the denominator as $(x - 1)(x + 2)$ and the decomposition into partial fractions with undetermined coefficients.

$$\frac{10 - x}{(x - 1)(x + 2)} = \frac{A}{x - 1} + \frac{B}{x + 2}.$$

We clear the fractions by multiplying by a common denominator $(x - 1)(x + 2)$ (the denominator of the left hand side).

$$10 - x = A(x + 2) + B(x - 1).$$

This equality is supposed to be true for all x . In particular it must be true for $x = 1$ and $x = -2$ which are the roots of linear factors. If we substitute $x = 1$, we get

$$10 - 1 = A(1 + 2) + B(1 - 1) \Rightarrow 9 = 3A + 0 \Rightarrow A = 3,$$

and by substitution $x = -2$ we get

$$10 - (-2) = A(-2 + 2) + B(-2 - 1) \Rightarrow 12 = 0 - 3B \Rightarrow B = -4.$$





We have all coefficients, so the integral is

$$\int \frac{10-x}{(x-1)(x+2)} dx = \int \frac{3}{x-1} - \frac{4}{x+2} dx = 3 \ln|x-1| - 4 \ln|x+2| + C.$$

To do this we first split it up into two integrals and then used the substitutions $t = x - 1$, $s = x + 2$ on the integrals. □

Example 101. Evaluate the integral $\int \frac{-x^2 + 5x - 4}{x(x^2 - 2x + 2)} dx$.

Solution: The denominator is already factored because the quadratic term in the denominator has no real roots. The partial fraction decomposition has the form

$$\frac{-x^2 + 5x - 4}{x(x^2 - 2x + 2)} = \frac{A}{x} + \frac{Bx + C}{x^2 - 2x + 2}.$$

We clear the fractions by multiplying by a common denominator.

$$-x^2 + 5x - 4 = A(x^2 - 2x + 2) + (Bx + C)x.$$

We multiply out the right side of this equality and collect all the like powers of x together.

$$-x^2 + 5x - 4 = (A + B)x^2 + (-2A + C)x + 2A.$$

We get the following system of equations

$$\begin{array}{rcl} x^2 : & -1 & = A + B, \\ x^1 : & 5 & = -2A + C, \\ x^0 : & -4 & = 2A, \end{array}$$

which is easy to solve and we have $A = -2$, $B = 1$ and $C = 1$. Thus

$$\int \frac{-x^2 + 5x - 4}{x(x^2 - 2x + 2)} dx = \int \frac{-2}{x} dx + \int \frac{x + 1}{x^2 - 2x + 2} dx.$$

The first integral leads to $-2 \ln x$. The second integral requires several steps. The quadratic term in the denominator has no real roots. We need to create the derivative of $x^2 - 2x + 2$ in the numerator. Since $(x^2 - 2x + 2)' = 2x - 2$, we have

$$x + 1 = \frac{1}{2} \cdot (2x + 2) = \frac{1}{2} \cdot (2x - 2 + 2 + 2) = \frac{1}{2} \cdot (2x - 2 + 4) = \frac{1}{2} \cdot (2x - 2) + 2.$$

We rewrite the integrand

$$\frac{x + 1}{x^2 - 2x + 2} = \frac{1}{2} \frac{2x - 2}{x^2 - 2x + 2} + \frac{2}{x^2 - 2x + 2}$$

and integrate the sum term by term and factor out constants:

$$\int \frac{x + 1}{x^2 - 2x + 2} dx = \frac{1}{2} \int \frac{2x - 2}{x^2 - 2x + 2} dx + 2 \int \frac{1}{x^2 - 2x + 2} dx.$$





For the integrand $\frac{2x-2}{x^2-2x+2}$ we substitute $t = x^2 - 2x + 2$ and $dt = (2x-2)dx$. For the integrand $\frac{1}{x^2-2x+2}$ we complete the square in the denominator and then use substitution.

$$\begin{aligned}\int \frac{x+1}{x^2-2x+2} dx &= \frac{1}{2} \int \frac{1}{t} dt + 2 \int \frac{1}{(x-1)^2+1} dx = \left| \begin{matrix} s = x-1 \\ ds = dx \end{matrix} \right| = \\ &= \frac{1}{2} \ln |t| + 2 \int \frac{1}{s^2+1} ds = \frac{1}{2} \ln |t| + 2 \cdot \arctan s + C = \\ &= \frac{1}{2} \ln |x^2 - 2x + 2| + 2 \arctan(x-1) + C.\end{aligned}$$

Now, we have

$$\int \frac{-x^2 + 5x - 4}{x(x^2 - 2x + 2)} dx = -2 \ln x + \frac{1}{2} \ln |x^2 - 2x + 2| + 2 \arctan(x-1) + C.$$

□



5 Definite Integral

In this section we will practise evaluating definite integral and use definite integral in geometric and physical applications.

5.1 Motivation

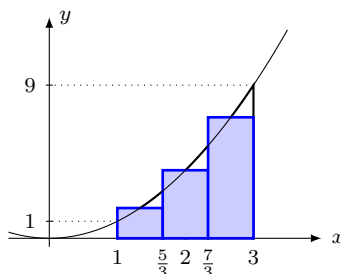
According to the lecture presentation slides the definite integral of f from a to b is given by

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}),$$

where ξ_i is any point in $[x_{i-1}, x_i]$. If f is continuous on the closed interval $[a, b]$, this limit exists and is unique, it does not depend on the choice of ξ_i , $i = 1, \dots, n$.

Example 102. Estimate the area of the region between $f(x) = x^2$ and the x -axis on $[1, 3]$. Choose $n = 3$ and use both left-endpoints and right-endpoints of the subintervals for the height of the rectangles.

Solution: For $n = 3$ let us consider the regular partition $D_3 = \{1, \frac{5}{3}, \frac{7}{3}, 3\}$.



Using a left-endpoint approximation, the area of the region between the graph of f and the x -axis is approximately

$$L_3(f, D) = \sum_{i=1}^3 f(x_{i-1}) \cdot (x_{i+1} - x_i) = (1)^2 \cdot \frac{2}{3} + \left(\frac{5}{3}\right)^2 \cdot \frac{2}{3} + \left(\frac{7}{3}\right)^2 \cdot \frac{2}{3} \doteq 6.14815.$$

Because the function f is increasing over the interval $[1, 3]$, it is so called a lower sum and underestimate. Using a right-endpoint approximation, the area of the region between the graph of f and the x -axis is approximately

$$R_3(f, D) = \sum_{i=1}^3 f(x_i) \cdot (x_{i+1} - x_i) = \left(\frac{5}{3}\right)^2 \cdot \frac{2}{3} + \left(\frac{7}{3}\right)^2 \cdot \frac{2}{3} + (3)^2 \cdot \frac{2}{3} \doteq 11.4815.$$



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Because the function f is increasing over the interval $[1, 3]$, it is so called an upper sum and overestimate. Compare these results with the lecture presentation slides, Example 5.4.

□

5.2 The Fundamental Theorem of Calculus

We can evaluate a definite integral of continuous function f on $[a, b]$ using antiderivative

$$\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a).$$

Example 103. Evaluate the integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2 x} dx$.

Solution:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2 x} dx = [-\cot x]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -\cot \frac{\pi}{2} + \cot \frac{\pi}{4} = 0 + 1 = 1.$$

□

5.3 Integration by Parts for Definite Integrals

Compared to indefinite integral the integration-by-parts formula for definite integral has an additional step, evaluating the integral at the upper and lower limits of integration.

Example 104. Evaluate the integral $\int_0^1 x \cdot 2^x dx$.

Solution: The integration by parts formula gives

$$\begin{aligned} \int_0^1 x \cdot 2^x dx &= \left| \begin{array}{ll} u = x & v' = 2^x \\ u' = 1 & v = \frac{2^x}{\ln 2} \end{array} \right| = \left[x \cdot \frac{2^x}{\ln 2} \right]_0^1 - \int_0^1 \frac{2^x}{\ln 2} dx = \\ &= \frac{2}{\ln 2} - 0 - \frac{1}{\ln 2} \cdot \left[\frac{2^x}{\ln 2} \right]_0^1 = \frac{2}{\ln 2} - \frac{1}{\ln 2} \cdot \left(\frac{2}{\ln 2} - \frac{1}{\ln 2} \right) = \frac{2}{\ln 2} - \frac{1}{\ln^2 2}. \end{aligned}$$

□

Example 105. Evaluate the integral $\int_{-1}^1 \arctan x dx$.

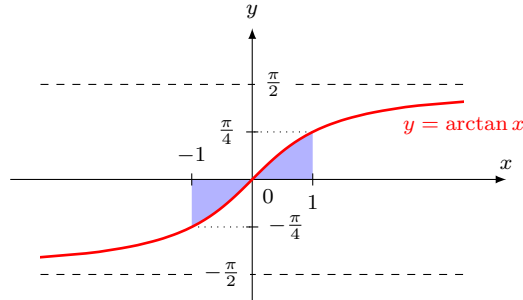
Solution: The integration by parts formula gives

$$\begin{aligned} \int_{-1}^1 \arctan x dx &= \left| \begin{array}{ll} u = \arctan x & v' = 1 \\ u' = \frac{1}{1+x^2} & v = x \end{array} \right| = [x \cdot \arctan x]_{-1}^1 - \int_{-1}^1 \frac{x}{1+x^2} dx = \\ &= 1 \cdot \arctan 1 - (-1 \cdot \arctan(-1)) - \frac{1}{2} \cdot \int_{-1}^1 \frac{2x}{1+x^2} dx = 0 - \frac{1}{2} \cdot [\ln(1+x^2)]_{-1}^1 = \\ &= -\frac{1}{2} \cdot [\ln(1+1^2) - \ln(1+(-1)^2)] = 0. \end{aligned}$$





The graph of $f(x) = \arctan x$ is shown in the following figure. There is a symmetry about the origin, the function f is an odd function. The region bounded by the graph of the function f and the x -axis is above the x -axis over $[0, 1]$ and below the x -axis over $[-1, 0]$. The areas of these two parts are equal. Therefore, the result of given integral is 0.



□

5.4 Substitution Rule for Definite Integrals

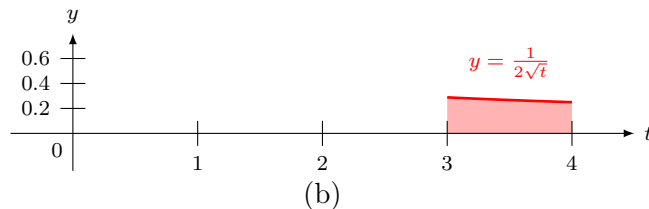
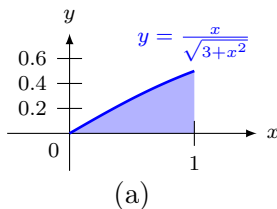
Using substitution to evaluate a definite integral we eliminate all the x 's in the integral including the limits of the integral.

Example 106. Evaluate the integral $\int_0^1 \frac{x}{\sqrt{3+x^2}} dx$.

Solution: Let us put $t = 3 + x^2$, $dt = 2x dx$. We need to change the limits of the integral. If $x = 0$ then $t = 3 + 0^2 = 3$. For $x = 1$ we have $t = 3 + 1^2 = 4$.

$$\begin{aligned} \int_0^1 \frac{x}{\sqrt{3+x^2}} dx &= \left. \begin{array}{l} t = 3 + x^2 \\ dt = 2x dx \\ \frac{1}{2} dt = x dx \\ x = 0 \rightarrow t = 3 \\ x = 1 \rightarrow t = 4 \end{array} \right| = \frac{1}{2} \int_3^4 \frac{1}{\sqrt{t}} dt = \frac{1}{2} \int_3^4 t^{-\frac{1}{2}} dt = \frac{1}{2} \left[\frac{t^{\frac{1}{2}}}{\frac{1}{2}} \right]_3^4 = \\ &= \frac{1}{2} \cdot 2 \cdot \left[\sqrt{t} \right]_3^4 = (\sqrt{4} - \sqrt{3}) = 2 - \sqrt{3}. \end{aligned}$$

The following figure shows the two regions. There is the region defined by the original integrand in the part (a) and the region defined by the new integrand in (b). These regions have the same area.





□

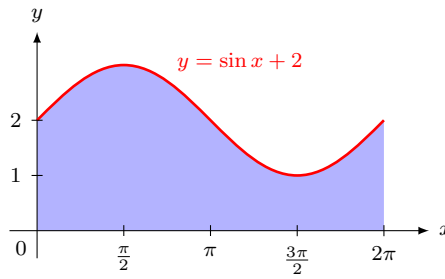
5.5 Applications of Definite Integral in Geometry

Let f and g be continuous functions with $f(x) \geq g(x)$ for $x \in [a, b]$. The area of the region bounded by the graphs of these functions and the lines $x = a$ and $x = b$ is

$$A = \int_a^b f(x) - g(x) \, dx.$$

Example 107. Determine the area of the region bounded by the graphs of $f(x) = \sin x + 2$, $g(x) = 0$, $x = 0$ and $x = 2\pi$.

Solution: Let us sketch the graph of f and the region in question.



The upper boundary of the region is given by the graph of the function f and the lower boundary of the region is given by the graph of the function $g(x) = 0$. The area of the region is the value of the integral

$$\begin{aligned} \int_0^{2\pi} (f(x) - g(x)) \, dx &= \int_0^{2\pi} (\sin x + 2 - 0) \, dx = [-\cos x + 2x]_0^{2\pi} = \\ &= -\cos 2\pi + 2 \cdot 2\pi - (-\cos 0 + 0) = -1 + 4\pi + 1 = 4\pi \text{ units}^2. \end{aligned}$$

□

Now, let us assume, that the graph of a non-negative function is represented by parametric equations $x = x(t)$, $y = y(t)$, $\alpha \leq t \leq \beta$. If y is continuous function on $[\alpha, \beta]$ and x is differentiable on $[\alpha, \beta]$, then the area under this graph is given by

$$A = \int_{\alpha}^{\beta} |y(t) \cdot x'(t)| \, dt.$$

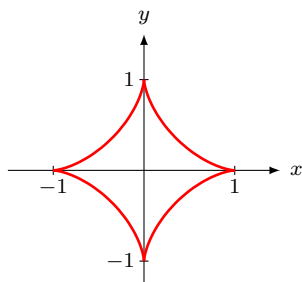
Example 108. Determine the area of the region bounded by the curve of the astroid defined by the equations $x(t) = \cos^3 t$, $y(t) = \sin^3 t$ for $0 \leq t \leq 2\pi$.

Solution: We want to find the area of the region in the following graph.



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We can divide this region into four parts and it is sufficient to find the area of the region for $t \in [0, \pi/2]$ and multiply it by 4. Then we have

$$\begin{aligned} A &= 4 \cdot \int_0^{\pi/2} |y(t) \cdot x'(t)| \, dt = 4 \cdot \int_0^{\pi/2} |\sin^3 t \cdot (-3 \sin t \cdot \cos^2 t)| \, dt = 12 \cdot \int_0^{\pi/2} \sin^4 t \cdot \cos^2 t \, dt = \\ &= 12 \cdot \int_0^{\pi/2} \sin^4 t \cdot (1 - \sin^2 t) \, dt = 12 \cdot \int_0^{\pi/2} \sin^4 t \, dt - 12 \cdot \int_0^{\pi/2} \sin^6 t \, dt. \end{aligned}$$

For integrals of this type, where are only even powers of $\sin x$ and $\cos x$, we use the identities

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

and simplify the integrand.

$$\begin{aligned} A &= 12 \cdot \int_0^{\pi/2} (\sin^2 t)^2 \, dt - 12 \cdot \int_0^{\pi/2} (\sin^2 t)^3 \, dt = 12 \cdot \int_0^{\pi/2} \left(\frac{1 - \cos 2t}{2} \right)^2 \, dt - 12 \cdot \int_0^{\pi/2} \left(\frac{1 - \cos 2t}{2} \right)^3 \, dt = \\ &= 3 \cdot \int_0^{\pi/2} (1 - 2 \cos 2t + \cos^2 2t) \, dt - \frac{3}{2} \cdot \int_0^{\pi/2} (1 - 3 \cos 2t + 3 \cos^2 2t - \cos^3 2t) \, dt = \\ &= \frac{3}{2} \cdot \int_0^{\pi/2} 1 \, dt - \frac{3}{2} \cdot \int_0^{\pi/2} \cos 2t \, dt - \frac{3}{2} \cdot \int_0^{\pi/2} \frac{1 + \cos 4t}{2} \, dt + \frac{3}{2} \cdot \int_0^{\pi/2} \cos 2t \cdot \cos^2 2t \, dt = \\ &= \frac{3}{2} \cdot [t]_0^{\pi/2} - \frac{3}{2} \cdot \left[\frac{\sin 2t}{2} \right]_0^{\pi/2} - \frac{3}{4} \cdot \left[t + \frac{\sin 4t}{4} \right]_0^{\pi/2} + \frac{3}{2} \cdot \int_0^{\pi/2} \cos 2t \cdot (1 - \sin^2 2t) \, dt = \\ &= \frac{3}{2} \cdot \frac{\pi}{2} - 0 - \frac{3}{4} \cdot \frac{\pi}{2} + \frac{3}{2} \cdot \frac{1}{2} \cdot \left[\sin 2t - \frac{\sin^3 2t}{3} \right]_0^{\pi/2} = \frac{3\pi}{4} - \frac{3\pi}{8} + 0 = \frac{3\pi}{8} \text{ units}^2. \end{aligned}$$

For the integrand $\cos 2t$ we have used substitution $s = 2t$, for the integrand $\cos 4t$ we have used substitution $s = 4t$ and for the integrand $\cos 2t \cdot (1 - \sin^2 2t)$ we have used substitution $s = \sin 2t$. \square

If f has a continuous derivative on $[a, b]$, then the length of its graph from the point $[a, f(a)]$ to the point $[b, f(b)]$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx.$$





Example 109. Determine the length of the curve $f(x) = \frac{x^2}{4} - \frac{1}{2} \ln x$ on $[1, e]$.

Solution: Let us compute the derivative of f and its square.

$$f'(x) = \left(\frac{x^2}{4} - \frac{1}{2} \ln x \right)' = \frac{x}{2} - \frac{1}{2x}, \quad (f'(x))^2 = \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}.$$

The curve length is

$$\begin{aligned} L &= \int_a^b \sqrt{1 + (f'(x))^2} dx = \int_1^e \sqrt{\frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2}} dx = \int_1^e \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2} dx = \\ &= \int_1^e \left(\frac{x}{2} + \frac{1}{2x}\right) dx = \left[\frac{x^2}{4}\right]_1^e + \left[\frac{1}{2} \ln |x|\right]_1^e = \frac{e^2}{4} - \frac{1}{4} + \frac{1}{2} \ln e - \frac{1}{2} \ln 1 = \frac{e^2}{4} + \frac{1}{4}. \end{aligned}$$

□

Now, let us assume, that the plane curve is represented by parametric equations $x = x(t)$, $y = y(t)$, $\alpha \leq t \leq \beta$. If x and y differentiable functions for $t \in [\alpha, \beta]$, then the arc length of this curve is given by

$$L = \int_{\alpha}^{\beta} \sqrt{(x'(t))^2 + (y'(t))^2} dt.$$

Example 110. Determine the length of the curve of the astroid defined by the equations $x(t) = \cos^3 t$, $y(t) = \sin^3 t$ for $0 \leq t \leq 2\pi$.

Solution: We have $x'(t) = -3\cos^2 t \sin t$, $y'(t) = 3\sin^2 t \cos t$. The arc length is

$$\begin{aligned} L &= 4 \cdot \int_0^{\frac{\pi}{2}} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} dt = 4 \cdot \int_0^{\frac{\pi}{2}} \sqrt{9\cos^2 t \sin^2 t \cdot (\cos^2 t + \sin^2 t)} dt = \\ &= 12 \cdot \int_0^{\frac{\pi}{2}} \sin t \cos t dt = \left[\begin{array}{l} s = \sin t \\ ds = \cos t dt \\ t = 0 \rightarrow s = 0 \\ t = \frac{\pi}{2} \rightarrow s = 1 \end{array} \right] = 12 \cdot \int_0^1 s ds = 12 \cdot \left[\frac{s^2}{2} \right]_0^1 = 6. \end{aligned}$$

□

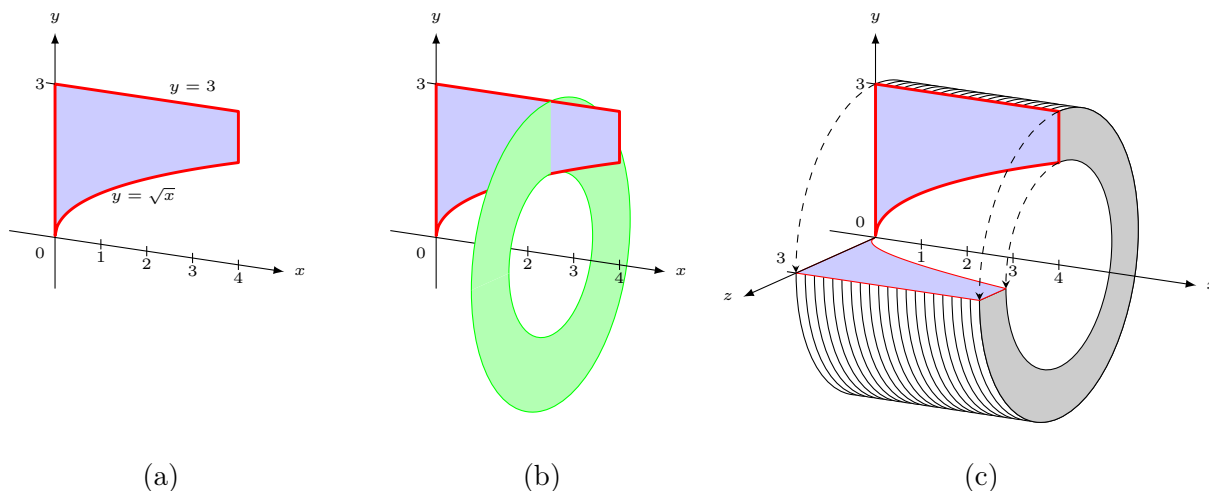
Let f and g be continuous and non-negative functions on an interval $[a, b]$ such that $f(x) \geq g(x)$ for $x \in [a, b]$. Then the volume of the solid of revolution generated by revolving a region bounded above by the graph of a function f and bounded below by the graph of a function g on an interval $[a, b]$ about the x -axis is given by

$$V = \pi \int_a^b (f^2(x) - g^2(x)) dx.$$

Example 111. Determine the volume of the solid obtained by rotating the region between the graphs of $y = 3$ and $y = \sqrt{x}$ over the interval $[0, 4]$ about the x -axis.

Solution: First we sketch the region (in the figure (a)). Rotating about the x -axis will produce cross sections in the shape of washers (in the figure (b)). The complete solid is shown in the figure (c).





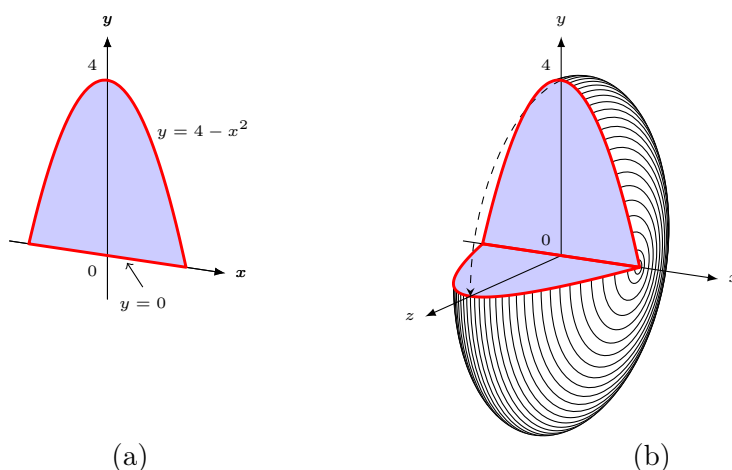
The volume of the solid is

$$\begin{aligned}
 V &= \pi \int_a^b (f^2(x) - g^2(x)) \, dx = \pi \int_0^4 ((3)^2 - (\sqrt{x})^2) \, dx = \pi \int_0^4 (9 - x) \, dx = \\
 &= \pi \left[9x - \frac{x^2}{2} \right]_0^4 = \pi \left(9 \cdot 4 - \frac{4^2}{2} \right) = \pi (36 - 8) = 24\pi.
 \end{aligned}$$

□

Example 112. Determine the volume of the solid obtained by rotating the region between the graphs of $y = 4 - x^2$ and $y = 0$ about the x -axis.

Solution: Graphing the region between two curves and then revolving the region about the line $y = 0$, we see the solid in the following figure.





The limits of integration are the points of the intersection of the graph of $y = 4 - x^2$ and the x -axis:

$$x^2 - 4 = 0 \Rightarrow x = \pm 2.$$

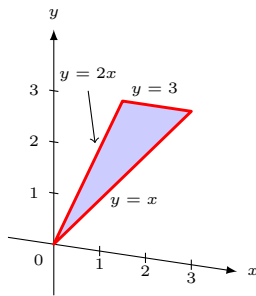
The volume of the solid is

$$\begin{aligned} V &= \pi \int_a^b (f^2(x) - g^2(x)) \, dx = \pi \int_{-2}^2 ((4 - x^2)^2 - (0)^2) \, dx = 2\pi \int_0^2 (16 - 8x^2 + x^4) \, dx = \\ &= 2\pi \left[16x - 8\frac{x^3}{3} + \frac{x^5}{5} \right]_0^2 = 2\pi \left(16 \cdot 2 - 8\frac{2^3}{3} + \frac{2^5}{5} \right) = 2\pi \left(32 - \frac{64}{3} + \frac{32}{5} \right) = \frac{512\pi}{15}. \end{aligned}$$

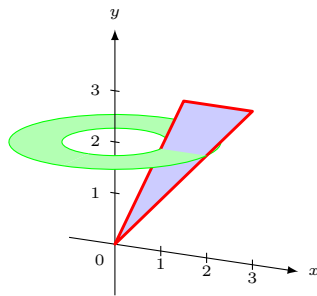
□

Example 113. Determine the volume of the solid obtained by rotating the region bounded by $y = 2x$, $y = x$ and $y = 3$ about the y -axis.

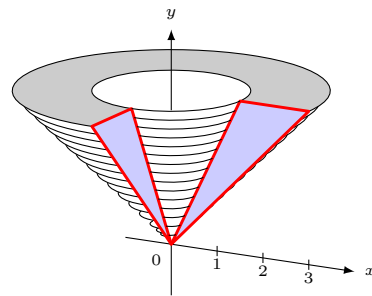
Solution: First we graph the three curves to see the region (a), rotating about the y -axis we can see the solid (c) and its cross section (b).



(a)



(b)



(c)

We need to convert given functions into the functions of y and convert x -bounds to y -bounds and then use the following formula

$$V = \pi \int_c^d (u^2(y) - v^2(y)) \, dy,$$

where u, v are continuous, non-negative functions with $v(y) \leq u(y)$ for $y \in [c, d]$. The outside radius $u(y)$ is formed by the line $y = x$, we solve it for x and get $x = y = u(y)$. We also need to solve $y = 2x$ for x , hence $x = \frac{y}{2} = v(y)$. Using a definite integral from $y = 0$ to $y = 3$ the volume of the solid is

$$\begin{aligned} V &= \pi \int_c^d (u^2(y) - v^2(y)) \, dy = \pi \int_0^3 \left((y)^2 - \left(\frac{y}{2} \right)^2 \right) \, dy = \pi \int_0^3 \left(y^2 - \frac{y^2}{4} \right) \, dy = \\ &= \frac{3\pi}{4} \int_0^3 y^2 \, dy = \frac{3\pi}{4} \left[\frac{y^3}{3} \right]_0^3 = \frac{3\pi}{4} \cdot \frac{3^3}{3} = \frac{27\pi}{4}. \end{aligned}$$

□



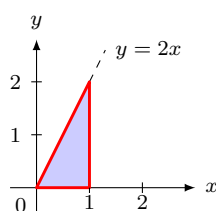


Let f be a nonnegative function over the interval $[a, b]$ with a continuous derivative on $[a, b]$. Then the surface area of the solid of revolution generated by revolving a region bounded above by the graph of a function f and bounded below by the x -axis on an interval $[a, b]$ about the x -axis is given by

$$\text{Surface Area} = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx.$$

Example 114. Determine the surface area of the solid obtained by rotating the region between the graph of $y = 2x$ and the x -axis over the interval $[0, 1]$ about the x -axis.

Solution: First we sketch the region.



Rotating this region about the x -axis will produce a cone with the height equalled to 1 and with the radius of the base equalled to 2. The surface area SA is

$$SA = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx = 2\pi \int_0^1 2x \sqrt{1 + (2)^2} dx = 4\sqrt{5}\pi \left[\frac{x^2}{2} \right]_0^1 = 4\sqrt{5}\pi \cdot \frac{1}{2} = 2\sqrt{5}\pi.$$

□

5.6 Applications of Definite Integral in Physics

5.6.1 Work Done by a Force

Let $F(x)$ be a continuous function describing a variable force that moves an object in a positive direction along the x -axis from point a to point b . The total work done by the force F is

$$W = \int_a^b F(x) dx.$$

The units of the integral are joules if F is in newtons and x is in meters.

Example 115. Suppose that a climbing rope hangs over the side of a cliff. The rope is 50 meters long and has a mass of 0.6 kg per meter. How much work is required to pull up the rope?

Solution: If x is the amount of the rope pulled in, the amount of the rope still hanging is $50 - x$. This part of the rope has a mass $0.6 \cdot (50 - x)$ kg. The variable force function is

$$F(x) = 9.8 \cdot 0.6 \cdot (50 - x) = 0.588(50 - x),$$





where 9.8 m/s^2 is the acceleration of gravity. The total work done by the force F to pull up the rope is

$$W = \int_0^{50} 0.588(50 - x)dx = 735 \text{ J.}$$

□

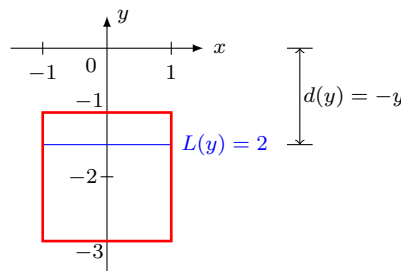
5.6.2 Fluid Forces

Let a vertically oriented plate be submerged in a fluid with weight-density w , where the top of the plate is at $y = b$ and the bottom is at $y = a$. Let $L(y)$ be the length of the plate measured from left to right along the surface of the plate at level y . Let $d(y)$ represent the distance between the surface of the fluid and the plate at y . Then the force exerted on the plate by the fluid is

$$F = \int_a^b w \cdot d(y) \cdot L(y) dy.$$

Example 116. A flat square plate with the base 2 m is submerged in water with a weight-density of 9807 N/m^3 . If the top of the plate is 1 m below the surface of the water, what is the total fluid force exerted against one side of the plate?

Solution: Let $y = 0$ represent the surface of the water. Then the top of the plate is at $y = -1$. We can center the square on the y -axis as shown in the following figure.



The length of the plate measured from left to right at level y is $L(y) = 2$. The depth of the plate at y is $d(y) = -y$. The total fluid force exerted against one side of the plate is

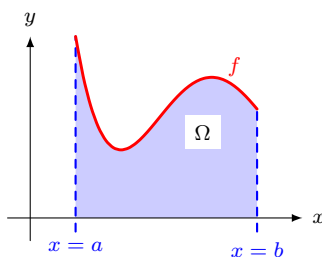
$$F = \int_a^b w \cdot d(y) \cdot L(y) dy = \int_{-3}^{-1} 9807 \cdot (-y) \cdot 2 dy = 78456 \text{ N.}$$

□

5.6.3 Center of Mass of a Thin Plate as a Region in the xy -Plane

Let f be a continuous and positive function on a closed interval $[a, b]$. Let us consider a region Ω bounded above by the graph of a function f , bounded below by the x -axis, bounded to the left by the vertical line $x = a$ and to the right by the vertical line $x = b$. Let $\rho(x)$ be the density of the thin plate for $x \in [a, b]$ represented by a two-dimensional region Ω in a plane.





The mass of the plate is

$$M = \int_a^b \varrho(x) f(x) dx.$$

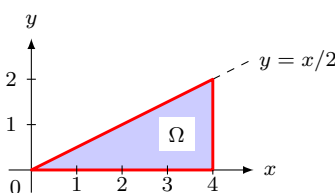
The values of

$$S_x = \frac{1}{2} \cdot \int_a^b \varrho(x) f^2(x) dx, \quad S_y = \int_a^b \varrho(x) x f(x) dx$$

are called the moments of the plate with respect to the x - and y -axes, respectively. The coordinates of the center of mass (the centroid) $T = [x_t, y_t]$ are then

$$x_t = \frac{S_y}{M}, \quad y_t = \frac{S_x}{M}.$$

Example 117. Let the triangular plate Ω shown in the figure below have a density of $\varrho = x^2$ at its points (x, y) . Find the centroid of the plate.



Solution: First, we calculate the total mass.

$$M = \int_a^b \varrho(x) f(x) dx = \int_0^4 x^2 \cdot \frac{x}{2} dx = \frac{1}{2} \cdot \int_0^4 x^3 dx = \frac{1}{2} \cdot \left[\frac{x^4}{4} \right]_0^4 = \frac{1}{2} \cdot \frac{1}{4} \cdot 4^4 = 32.$$

Next, we compute the moments.

$$S_x = \frac{1}{2} \cdot \int_a^b \varrho(x) f^2(x) dx = \frac{1}{2} \cdot \int_0^4 x^2 \cdot \frac{x^2}{4} dx = \frac{1}{8} \cdot \int_0^4 x^4 dx = \frac{1}{8} \cdot \left[\frac{x^5}{5} \right]_0^4 = \frac{1}{8} \cdot \frac{1}{5} \cdot 4^5 = \frac{128}{5},$$

$$S_y = \int_a^b \varrho(x) x f(x) dx = \int_0^4 x^2 \cdot x \cdot \frac{x}{2} dx = \frac{1}{2} \cdot \int_0^4 x^4 dx = \frac{1}{2} \cdot \left[\frac{x^5}{5} \right]_0^4 = \frac{1}{2} \cdot \frac{1}{5} \cdot 4^5 = \frac{512}{5}.$$

The coordinates of the centroid are

$$x_t = \frac{S_y}{M} = \frac{\frac{512}{5}}{32} = \frac{16}{5}, \quad y_t = \frac{S_x}{M} = \frac{\frac{128}{5}}{32} = \frac{4}{5}.$$

□

